A Review of the Frequency Estimation and Tracking Problems

P.J. Kootsookos
CRC for Robust and Adaptive Systems
DSTO, Salisbury Site
Frequency Estimation and Tracking Project

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Abstract

This report presents a concise review of some frequency estimation and frequency tracking problems. In particular, the report focusses on aspects of these problems which have been addressed by members of the Frequency Tracking and Estimation project of the Centre for Robust and Adaptive Systems.

The report is divided into four parts: problem specification and discussion, associated problems, frequency estimation algorithms and frequency tracking algorithms.

Part I begins with a definition of the various frequency estimation and tracking problems. Practical examples of where each problem may arise are given. A comparison is made between the frequency estimation and tracking problems.

In Part II, block frequency estimation algorithms, fast block frequency estimation algorithms and notch filtering techniques for frequency estimation are dealt with.

Frequency tracking algorithms are examined in Part III.

Part IV of this report examines various problems associated with frequency estimation. Associated problems include Cramér-Rao lower bounds, theoretical algorithm performance, frequency resolution, use of the analytic signal and model order selection.

Contents

Part I

Frequency Estimation and Tracking Problems

Frequency estimation and tracking problems, algorithms and related topics are discussed in this report. The aim is to present a concise sketch of these problems, describe current techniques and indicate loose ends.

In this Part, several estimation and tracking problems are specified and examined. The two problems to be given most scrutiny in the remainder of the report are single-tone frequency estimation and single-tone frequency tracking. We discuss the reasons for this emphasis.

Part II of this report describes several block-processing frequency estimation algorithms, starting with the standard Gaussian, white noise maximum likelihood approach. Other approaches discussed include the periodogram maximiser, Fourier coefficient interpolation and and sample covariance techniques. Details are then given for some fast block-processing algorithms which are of interest because of their computational simplicity.

The Part concludes with the examination of several notch filtering techniques for frequency estimation. The last of these techniques, that of Nehorai and Porat [1986] may be used as a frequency tracking algorithm, and so this naturally leads on to Part III where several frequency tracking algorithms are examined.

Further issues such as use of the analytic signal, frequency resolution and model order reduction are canvassed in Part IV.

Rather than list the possible future directions for research on these topics out of context, areas where further work is required are indicated throughout the report.

1 The Frequency Estimation Problem

There are three main parameter estimation problems which involve frequency estimation:

- Single tone frequency estimation: where the signal is a single, constant-frequency sinusoid. This is the simplest frequency estimation problem.
- Multi-harmonic frequency estimation: where the signal is composed of the sum of harmonically related sinusoids. This case is important because rotational or periodic phenomena rarely generate sinusoidal waveforms.
- Multi-tone frequency estimation: where there are several tones of unrelated frequency present. This problem occurs in the analysis of signals containing emissions from more than one target.

Each problem assumes a different signal model is to be fitted to measured data. Due to the large amount of literature available, this discussion will be confined to the first problem, that of estimating the parameters of a single tone in noise.

1.1 Single Tone Frequency Estimation

The single tone in noise parameter estimation problem is defined as follows.

Problem 1: Single Tone in Noise

Let $\{y_t\}$ be generated by the model

$$y_t = \mu + \rho \cos\left(\omega_0(t-\nu) + \phi\right) + \varepsilon_t \tag{1}$$

where μ is the mean value, ρ is the signal amplitude, ω_0 is the frequency of the signal, $\nu = \frac{T-1}{2}$, ϕ is the initial phase and ε_t is some zero-mean random noise sequence with variance σ^2.

The single tone in noise problem is then to estimate the real-valued parameters μ, ρ, ω_0, ϕ and σ^2 given only the measurements $\{y_t : t = 0 \ldots T-1\}$. □

A related problem is estimation of the parameters in the following signal model:

$$z_t = \mu' + \rho \exp i\left(\omega_0(t-\nu) + \phi\right) + \varepsilon'_t \tag{2}$$

where the parameters are as before, except that both μ' and ε'_t are complex-valued. Rife and Boorstyn [1974] examined this case. For the majority of this report, however, we shall confine our discussion to the real signal model y_t of (1).

Remark 1: *The signal model (2) is sometimes referred to as the* analytic signal *(Ville [1948]) model.* □

In all of these problems, algorithm selection depends on whether the sample size T is fixed or increasing. If T is fixed, block-processing algorithms are considered. For T increasing, on-line algorithms are of interest.

1.2 Multi-harmonic Frequency Estimation

Where frequency information is to be gleaned from acoustic sources such as rotating machinery, non-linear effects within the generating system often give rise to harmonics and sub-harmonics of the fundamental mode of interest. In these situations, (1) does not model the physical situation well, so a signal model which accounts for the added harmonics should be used.

Such a signal model is given below.

Problem 2: Harmonically-related Tones in Noise

Let $\{y_t\}$ be a multi-harmonic signal modelled by

$$y_t = \mu + \sum_{j=1}^{p} \rho_j \cos\left(j\omega_0(t-\nu) + \phi_j\right) + \varepsilon_t \tag{3}$$

where μ, ν, ε_t and σ^2 are defined as in Problem 1 and ω_0 is now the fundamental frequency of the signal, ρ_j is the amplitude of the j^{th} harmonic and ϕ_j is the initial phase of the j^{th} harmonic.

The multiharmonic-related tones in noise problem is then to estimate the parameters μ, ρ_j, ω_0, ϕ_j, σ^2 and p given only the measurements $\{y_t : t = 0 \ldots T-1\}$. □

The papers by Barrett and McMahon [1987], James, Anderson and Williamson [1991a] and James, Anderson and Williamson [1991b] examine the multiharmonic frequency estimation problem.

1.3 Multi-tone Frequency Estimation

In certain environments, several tonal sources of differing frequencies may be present in the one signal. While, in some cases, it may be possible to apply single-tone techniques in this situation, it is more desirable to account for the extra problem complexity by altering the signal model.

The multiple tones in noise problem is defined as follows.

Problem 3: Multiple Tones in Noise
Let $\{y_t\}$ be a multi-tonal signal modelled by

$$y_t = \mu + \sum_{j=1}^{p} \rho_j \cos\left(\omega_j(t-\nu) + \phi_j\right) + \varepsilon_t \tag{4}$$

where μ, ν, ε_t and σ^2 are defined as in Problem 1 and ω_j is now the frequency of the j^{th} signal component, ρ_j is the amplitude of the j^{th} tone and ϕ_j is the initial phase of the j^{th} tone.

The multiple tones in noise problem is then to estimate the parameters μ, ρ_j, ω_j, ϕ_j, σ^2 and p given only the measurements $\{y_t : t = 0 \ldots T-1\}$. □

Hannan [1973] has examined this case and Rife and Boorstyn [1976] have examined the equivalent complex signal model problem.

1.4 Frequency Estimation

In this report, we shall only examine in detail algorithms for the frequency estimation problem. The main reason for this is that the added complexity introduced by signal models (3) and (4) obfuscates some of the key issues by either introducing new problems or by increasing the dimensionality of the problem.

For instance, use of model (4) instead of (1) means that model order selection now becomes an issue. We shall discuss this and other issues further in Part IV.

2 The Frequency Tracking Problem

The frequency tracking problem is somewhat more complicated than the estimation problem. Three physical situations where frequency tracking is of interest are

1. decoding digital information from a frequency-shift keyed bit stream,
2. in demodulation of an FM radio signal and
3. tracking the revolutions per minute of the engine of a manœuvring vessel via acoustic data.

Each of these three problems may be described by the following general problem statement.

Problem 4: Frequency Tracking
Let $\{y_t\}$ be modelled by

$$y_t = \mu + \rho \cos\left(\sum_{k=0}^{t} \omega_k + \phi\right) + \varepsilon_t \tag{5}$$

where ω_k is called the <u>*instantaneous frequency*</u> *of the signal and μ, ρ, ν, ϕ, ε_t and σ^2 are as defined in Problem 1.*

The frequency tracking problem is then to estimate the signal parameters μ, ρ, ϕ and σ^2 and the sequence $\{\omega_k\}$ given only the measurements $\{y_t : t = 0 \ldots T-1\}$. □

Boashash [1992a] gives a general discussion of the problem of instantaneous frequency estimation.

Remark 2: *The most notable difference between this problem and Problems 1 to 3 is that the estimation of μ, ρ, ϕ and σ^2 is a* parameter *estimation problem, whereas estimation of the sequence $\{\omega_k : k = 0 \ldots T-1\}$ is a* state *estimation problem.* □

Remark 3: *The definition of this tracking problem does not immediately suggest an appropriate error measure. In parameter estimation problems it is common to use the least square error criterion. However, in tracking problems, concepts such as 'loss of track' or 'escape time' may be of more importance.* □

The escape time for the phase locked loop frequency tracker has been examined by Dupuis and Kushner [1987].

The three physical situations above may be examined by suitable selection of various parameters in Problem 4. Because of this different parameter selection, some algorithms may be more appropriate to apply to certain problems than others.

2.1 Demodulation of Digital Signals

For example, when a frequency-shift keyed bit stream is to be decoded, ω_k will be at known constant values for some duration, which is also perhaps known. While the constant frequencies are known, the change from frequency to frequency is generally stochastic.

Remark 4: *In this situation, hidden Markov models are probably the most appropriate approach to take due to the discrete nature of process of interest.* □

2.2 FM Demodulation

For the FM demodulation problem, set

$$\omega_k = \omega_c + \lambda_k \quad \text{with} \quad \omega_c \gg \lambda_k$$

where ω_c is the carrier frequency and λ_k is the speech signal to be demodulated. The frequency variation λ_k may possibly be modelled as an auto-regressive (or linear predictive) process.

In these cases, the noise is generally fairly small. For instance, because of the effect of threshold which is characterised by a severe degradation in reception below some noise level, the signal (or carrier) to noise ratio (SNR) for a commercial (stereo) FM station is usually required to be

$$10 \log_{10} \frac{\rho}{2\sigma^2} > 20\text{dB}.$$

Remark 5: *A major difference between this problem and that of digital signal demodulation is that here the frequency variation itself is modelled as being stochastic.* □

2.3 Tracking in High Noise

For the manœuvring vessel problem, ω_k will be slowly time-varying while, in general, the noise is large. For the underwater case, the operating signal to noise ratio is generally

$$10 \log_{10} \frac{\rho}{2\sigma^2} < -20\text{dB}.$$

Fortunately, because of *a priori* knowledge of the working environment, only a small band of known frequencies is generally of interest.

Remark 6: *The frequency tracking problem in this situation is thus further constrained to take place within a known 'gate' of possible frequencies.* □

Remark 7: *A problem that also needs attention is that of deciding whether or not an interesting signal is present given knowledge of the background noise. The issues are called track initiation or detection and track termination and they are of major concern.* □

2.4 Frequency Tracking

Part III of this report examines some particular frequency tracking algorithms, and some other aspects of the problem are examined in Part IV.

> Further Work 1: Methods for allowing *a priori* information about the instantaneous frequency law ω_k to be used should be examined.
>
> Further Work 2: Block estimators of frequency may be used as frequency trackers, provided the frequency variation is slow enough over the length of the data. Bounds on the performance of block and on-line techniques, for a given amount of frequency variation (specified either stochastically or deterministically), would allow direct comparisons between such techniques to be made.

Multi-harmonic and multi-tone extensions of Problem 4 may also be defined, however they are not discussed here as this will overly complicate the problem.

Part II
Frequency Estimation Algorithms

We shall categorise frequency estimation algorithms as follows:

1. block estimators, where the frequency estimate is obtained for a fixed sample size T in $O(T \log T)$ or more floating point operations,
2. fast block estimators, where the sample size is again fixed, but the number of operations required is $O(T)$, and
3. on-line estimators, which allow recursively updated frequency estimates to be generated.

This last class of estimators is of particular interest, because they may be more amenable to extension to the frequency tracking problem that the block-processing methods. The block-processing methods may only be used for tracking when it is known that the instantaneous frequency of the signal does not change significantly over known time periods.

1 Block Frequency Estimators

There are several approaches to frequency estimation, given a data set of fixed length T. We start by examining the maximum likelihood approach.

1.1 The Maximum Likelihood Estimator of Frequency

When ε_t is Gaussian with covariance $R_{\varepsilon\varepsilon}$ the maximum likelihood estimator (MLE) of frequency is simply the maximiser of the likelihood function given by

$$L(\theta) = \frac{1}{(2\pi)^{\frac{T}{2}}\sqrt{|R_{\varepsilon\varepsilon}|}} \exp(-\frac{(\boldsymbol{Y} - \hat{\boldsymbol{Y}}(\theta))^T R_{\varepsilon\varepsilon}^{-1} (\boldsymbol{Y} - \hat{\boldsymbol{Y}}(\theta))}{2})$$

$|A|$ is the determinant of the matrix A, $\hat{\boldsymbol{Y}}(\theta) = [\hat{y}_0 \; \hat{y}_1 \ldots \hat{y}_{T-1}]$ and $\theta' = [\mu \; \rho \; \omega \; \phi]$ with

$$\hat{y}_t = \mu + \rho \cos\left(\omega(t - \nu) + \phi\right)$$

with $\boldsymbol{Y}$ the vector of the noisy measurements. Equivalently, the maximiser of the log-likelihood function

$$\ell(\theta) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\ln(|R_{\varepsilon\varepsilon}|) - \frac{(\boldsymbol{Y} - \hat{\boldsymbol{Y}}(\theta))^T R_{\varepsilon\varepsilon}^{-1} (\boldsymbol{Y} - \hat{\boldsymbol{Y}}(\theta))}{2}. \tag{6}$$

may be used. If the ε_t are white, then $R_{\varepsilon\varepsilon} = I$ and maximising (6) is equivalent to minimising

$$\sum_{t=0}^{T-1} (y_t - \hat{y}_t)^2$$

which is just the least-square error between the data sequence $\{y_t\}$ and the model sequence $\{\hat{y}_t\}$.

These functions are usually maximized using a Newton method (Starer and Nehorai [1992]).

1.2 Approximate Maximum Likelihood Techniques

1.2.1 The Maximiser of the Periodogram

For the case when the complex signal model of (2) is used, and when ρ and ϕ are unknown and estimated using

$$\hat{\rho}e^{i\hat{\phi}} = \sum_{t=0}^{T-1} y_t e^{-i\hat{\omega}_0 t}$$

then the maximum likelihood frequency estimate is equivalent to setting

$$\begin{aligned}\hat{\omega}_P &= \max_{\omega} |\sum_{t=0}^{T-1} y_t e^{-i\omega t}|^2 \\ &= \max_{\omega} I_y(\omega)\end{aligned}$$

where I_y is the periodogram of the y_t.

For the real signal model of (1), then this equivalence is only asymptotically ($T \to \infty$) true. Hannan [1973] has examined this case and shown that,

$$T^{3/2}(\hat{\omega}_P - \omega_0) \sim N(0, 48\pi\rho^{-2} f_\varepsilon(\omega_0))$$

where $f_\varepsilon(\omega_0)$ is the spectral density of ε_t at the true frequency.

Due to the highly non-linear nature of this problem and the many local maxima of I_y, an initial frequency estimate at least as close as

$$|\hat{\omega}_\mathrm{i} - \omega_0| = O(T^{-\epsilon}) \;\; \text{with } \epsilon > 1$$

is needed (Quinn and Fernandes [1991]) if a Gauss-Newton function maximisation technique is used.

Remark 8: *Quinn and Fernandes [1991] noted that the length T fast Fourier transform of $\{y_t\}$ only yields an initial estimate with*

$$|\hat{\omega}_\mathrm{i} - \omega_0| = O(T^{-1})$$

which is not close enough to guarantee convergence of the Gauss-Newton iteration. □

1.3 Fourier Coefficient Techniques

There are several algorithms which rely on the phase and magnitude of the maximum modulus complex Fourier coefficient. Such techniques are useful because they are generally computationally simple. The most computationally intensive operations needed are a fast Fourier transform and a size T search to find the maximum modulus coefficient.

Remark 9: *The variance performance of such estimators approaches that of the maximum likelihood estimator of frequency, with significantly less computational cost.* □

1.3.1 Fourier Coefficient Interpolation Methods

Quinn [1992b] has devised the following Fourier coefficient interpolation frequency estimation algorithms.

1. Put $\hat{k}$ equal to the maximiser of the discrete-frequency periodogram

$$\hat{k} = \arg\max_k I_y(2\pi k/T).$$

2. Put

$$\begin{aligned} \hat{\alpha}_{+1} &= \Re\left(I_y(2\pi(\hat{k}+1)/T)/I_y(2\pi\hat{k})\right) \quad \delta_{+1} = -\hat{\alpha}_{+1}/(1-\hat{\alpha}_{+1}) \\ \hat{\alpha}_{-1} &= \Re\left(I_y(2\pi(\hat{k}-1)/T)I_y(2\pi\hat{k})\right) \quad \delta_{-1} = \hat{\alpha}_{-1}/(1-\hat{\alpha}_{-1}) \end{aligned}$$

   where $\Re(\cdot)$ indicates the real part of the argument.

3. To form the first estimator, $\hat{\omega}_{\text{FTI1}}$, if δ_{+1} and δ_{-1} are both positive, set $\delta = \delta_{+1}$. Otherwise set $\delta = \delta_{-1}$. Then

$$\hat{\omega}_{\text{FTI1}} = 2\pi(\hat{k}+\delta)/T.$$

4. To form the second estimator, $\hat{\omega}_{\text{FTI2}}$, put

$$\delta = (\delta_{+1}+\delta_{-1})/2 + \tau(\delta_{+1}^2) - \tau(\delta_{-1}^2)$$

   with

$$\tau(x) = \frac{1}{4}\log(3x^2+6x+1) - \frac{\sqrt{6}}{24}\log\left\{\frac{x+1-\sqrt{\frac{2}{3}}}{x+1+\sqrt{\frac{2}{3}}}\right\}$$

   then set

$$\hat{\omega}_{\text{FTI2}} = 2\pi(\hat{k}+\delta)/T.$$

Remark 10: *Quinn [1994] has developed central limit theorems for both estimators. As the second technique finds that nonlinear function of δ_{+1} and δ_{-1} which minimises the mean square error, it is not surprising that $\hat{\omega}_{\text{FTI2}}$ has the smaller mean square error.* □

1.3.2 The Generalised Phase Interpolation Estimator

When it is desired to use block-processing techniques to track a slow variation in frequency, McMahon and Barrett [1986] have described a frequency estimation technique which uses the maximal Fourier coefficients from adjacent (possibly overlapping) time-blocks.

The method proceeds as follows.

1. Form the length R discrete Fourier Transforms at frequency $\overline{\omega}$:

$$A = \sum_{t=0}^{R-1} y_t e^{i\overline{\omega}t} \qquad B = \sum_{t=0}^{R-1} y_{t+V} e^{i\overline{\omega}t}$$

   where $\overline{\omega}$ has been chosen to be close to the frequency to be estimated and V is the offset between the two time blocks.

2. Calculate

$$\delta = \arg(B) - \arg(A)$$

3. The estimator is then given by

$$\hat{\omega}_{\text{PIE}} = (\delta + 2\pi n)/V.$$

The phase ambiguity, $2\pi n/V$, may be resolved by choosing another offset V', recomputing $\hat{\omega}'_{\text{PIE}}$ and selecting n and n' so that $|\hat{\omega}_{\text{PIE}} - \hat{\omega}'_{\text{PIE}}|$ is as small as possible.

Remark 11: *McMahon and Barrett [1987] have extended this technique to the multi-tone case and have used this extension to perform multi-component frequency tracking.* □

1.4 Sample Covariance Methods

There are several frequency estimation methods available which rely on maximizing a signal spectral estimate other than the periodogram. Thus they may also be considered to be approximate maximum likelihood techniques. These spectral estimates are arrived at via the eigensystem decomposition of the $M \times M$ sample Toeplitz covariance matrix of the sequence $\{y_t\}$:

$$\hat{\mathrm{R}}_{yy} = \begin{bmatrix} \hat{r}_{yy}(0) & \hat{r}_{yy}(1) & \dots & \hat{r}_{yy}(T-1) \\ \hat{r}_{yy}(1) & \hat{r}_{yy}(0) & \ddots & \\ \vdots & \ddots & \ddots & \hat{r}_{yy}(1) \\ \hat{r}_{yy}(T-1) & & \hat{r}_{yy}(1) & \hat{r}_{yy}(0) \end{bmatrix} \tag{7}$$

where

$$\hat{r}_{yy}(m) = \frac{1}{T} \sum_{t=0}^{T-1-|m|} y_t \overline{y}_{t+m} \quad m = 0, \dots, M-1 \tag{8}$$

is the biased covariance estimator where $\overline{y}$ indicates the complex conjugation of y. The biased estimator is used to ensure that $\hat{\mathrm{R}}_{yy}$ is non-negative definite.

The Bartlett spectral estimate may be written

$$P_{\text{Bar}}(\omega) = v^*(\omega)\hat{\mathrm{R}}_{yy}v(\omega) \tag{9}$$

where $v(\omega) = [1 \exp(i\omega) \exp(i2\omega) \dots \exp(i(N-1)\omega)]^T$ and v^* indicates the complex conjugate transpose of the vector v. Note that if $M = T$ then this estimate is the same as the periodogram spectral estimator. Another popular spectral estimator is the minimum variance (Kay [1988, page 428]) spectral estimator given by

$$P_{\text{MV}}(\omega) = \frac{1}{v^*(\omega)\hat{\mathrm{R}}_{yy}^{-1}v(\omega)}. \tag{10}$$

The various signal subspace and noise subspace approaches substitute rank-reduced approximations or rescalings of the sample covariance matrix $\hat{\mathrm{R}}_{yy}$ (or its inverse), into (9) or (10). Both Kay [1988] and Marple [1987] give accounts of these types of spectral (and the associated frequency) estimators.

Given an eigenvector decomposition of the full-rank $\hat{\mathrm{R}}_{yy}$:

$$\hat{\mathrm{R}}_{yy} = \sum_{k=1}^{M} \lambda_k e_k e_k^*,$$

where $|\lambda_k| > |\lambda_{k+1}|$ and e_k is the eigenvector associated with λ_k, then the various methods use the following approximations of $\hat{\mathrm{R}}_{yy}$.

1.4.1 Signal Subspace Methods

The signal subspace methods use the eigenvectors of $\hat{\mathrm{R}}_{yy}$ corresponding to the p largest modulus eigenvalues (maximal eigenvectors) are used, where p is the number of sinusoids that are assumed to be present.

Minimum Variance The principle component minimum variance spectral estimator is formed by substitution of

$$\hat{R}_{\mathrm{MV}}^{-1} = \sum_{k=1}^{p} \frac{1}{\lambda_k} e_k e_k^*.$$

into (10).

Bartlett The principle component Bartlett spectral estimator is found by substituting

$$\hat{R}_{\mathrm{Bar}} = \frac{1}{M} \sum_{k=1}^{p} \lambda_k e_k e_k^*.$$

into (9).

1.4.2 Noise Subspace Methods

The noise subspace methods use the $M - p$ minimal eigenvectors of $\hat{\mathrm{R}}_{yy}$ to form an approximate inverse covariance matrix. Equation 10 is then maximised over ω to find the frequency estimates.

Pisarenko's Method Pisarenko harmonic decomposition (PHD) (Pisarenko [1973]) uses a sample autocovariance matrix of size $p + 1 \times p + 1$. The approximate inverse covariance matrix for this method is

$$\hat{R}_{\mathrm{Pis}}^{-1} = e_{p+1} e_{p+1}^*.$$

MUSIC The multiple signal classification (MUSIC) is a generalization of the Pisarenko approach which sets M greater than $p + 1$ so that

$$\hat{R}_{\mathrm{MUSIC}}^{-1} = \sum_{k=p+1}^{M} e_k e_k^*.$$

Choice of w_t	
Author	$w_t \quad t = 1, \ldots, T-1$
Lank, Reed and Pollon [1973]	$\frac{1}{T-1}$
Kay [1989]	$\frac{6t(T-t)}{T(T^2-1)}$
Lovell and Williamson [1992]	$\frac{6t(T-t)}{T(T^2-1)\lvert z_t\rvert\lvert z_{t-1}^*\rvert}$
Clarkson, Kootsookos and Quinn [1994]	$\frac{\sinh(T\theta)-\sinh(t\theta)-\sinh((T-t)\theta)}{(T-1)\sinh(T\theta)-2\sinh(\frac{1}{2}T\theta)\sinh\left[\frac{1}{2}(T-1)\theta\right]/\sinh(\frac{1}{2}\theta)}$ where $\theta = \ln\left(1+\frac{\sigma^2}{\rho^2}+\sqrt{\frac{\sigma^4}{\rho^4}+\frac{\sigma^2}{\rho^2}}\right)$

Table 1: Proposed choices of window function, w_t.

Remark 12: *Unfortunately, most of the estimators of frequency obtained using these methods are not asymptotically efficient. As the noise subspace methods tend to perform particularly badly (see, for example, Kay [1988, Figure 13.6] for a comparison) and because most are computationally intensive, these methods will not be closely examined in this report.* □

Remark 13: *For the (complex) single tone case, Pisarenko's method is of interest, because here $M = 2$ and all that is required is to find the roots of a quadratic polynomial. Hence, while this method is statistically inefficient it is very fast to calculate, and so the estimate it produces may be used to initialise other algorithms.* □

2 Fast Block Frequency Estimators

Many of the techniques discussed previously can be recast to work with the signal model of (2). Use of this model also allows another class of estimator to be defined: the weighted phase averaging frequency estimators.

These estimators are of interest because they may be performed in $O(T)$ floating point operations, and so may be considered for use instead of on-line methods in the frequency tracking problem.

The weighted linear predictor form of frequency estimator is given by

$$\hat{\omega}_0 = \arg\left(\sum_{t=1}^{T-1} w_t z_t z_{t-1}^*\right)$$

where w_t is some window function and $\arg(z)$ indicates finding the phase (argument) of the complex-valued z. Several different estimators may defined by different selection of the window function w_t. Table 1 gives a list of possible choices of w_t.

Lovell and Williamson [1992] introduced the signal-dependent form of w_t, while Clarkson, Kootsookos and Quinn [1994] showed how a class of windows, which depend on the SNR, may be generated. For high SNR, the SNR-dependent window is equivalent to Kay's window and for low SNR the SNR-dependent window becomes the rectangular window of Lank, Reed and Pollon [1973].

Other possible phase-related estimators are the weighted phase averagers of the form (Kay [1989])

$$\hat{\omega}_0 = \sum_{t=1}^{T-1} w_t \left[\arg\left(z_t\right) - \arg\left(z_{t-1}\right)\right].$$

Quinn [1992a] has shown that, where ε_t is Gaussian and w_t is not signal dependent, the weighted phase averaging form yields biased estimates of frequency. As a result, we will not examine these estimators. Clarkson [1992] has proposed a modification to Kay's estimator which adaptively attempts to circumvent this bias.

> Further Work 3: The statistical efficiency claimed of their estimator by Lovell and Williamson [1992] must be reconciled with the lack of statistical efficiency of the estimators examined in Clarkson, Kootsookos and Quinn [1994].

3 Notch Filtering Techniques

Given a $\{y_t\}$ of the form (1), then the y_t also satisfy

$$y_t - 2\cos(\omega_0) y_{t-1} + y_{t-2} = \varepsilon_t - 2\cos(\omega_0)\varepsilon_{t-1} + \varepsilon_{t-2}.$$

This ARMA(2,2) equation system involves pole-zero cancellations, and so its solution is not well defined.

There are several techniques which use this approach (Bhaskar Rao and Kung [1984]; Fernandes, Goodwin and de Souza [1987]; Nehorai and Porat [1986]; Quinn and Fernandes [1991]).

3.1 Off-line Filtering Techniques

While the techniques of Fernandes, Goodwin and de Souza [1987], Quinn and Fernandes [1991] are not on-line filtering techniques, they are examined here because they have much in common with the on-line methods of Hannan and Huang [1993] and Nehorai and Porat [1986].

3.1.1 The Technique of Fernandes, Goodwin and de Souza

The Fernandes, Goodwin and de Souza [1987] frequency estimation technique is described as follows.

1. Set $j = 1$ and let $h_{t,j} = \delta_t$ be the impulse response of a filter $H_j(\omega)$ where δ_t is the Kronecker delta.

2. Set $y^f_{t,j} = h_{t,j} * y_t$ for $t = 0, \ldots, T-1$ where $*$ denotes convolution.

3. Using $y^f_{t,j}$, obtain $\hat{a}_j$ an estimate of $2\cos(\omega_0)$ using equation error based least squares. If $|\hat{a}_j - \hat{a}_{j-1}|$ is small enough, set $\hat{\omega} = \cos^{-1}(\hat{a}_j/2)$ and terminate the algorithm. Otherwise, continue.

4. Construct a new filter, $H_{j+1}(\omega)$, the passband of which is highly likely to contain the true frequency. Set $h_{t,j+1}$ equal to the impulse response of this filter.

5. Increment j and go to step 2.

Remark 14: *The frequency estimate obtained via the equation error based least squares technique is biased, however Fernandes, Goodwin and de Souza [1987] bound this bias. This bound is used to set the bandwidth of the filter new filter $H_{j+1}(\omega)$.* □

For the technique to converge, the bandwidth of the filters $H_j(\omega)$ must be shown to decrease at each iteration. Conditions under which this is achieved are given in Lemma 4.2 of Fernandes, Goodwin and de Souza [1987]. An approach which also uses the idea of bandwidth contraction is described by Yakowitz [1992].

Remark 15: *The filters , $H_j(\omega)$, may be any bandpass filter with the appropriately selectable passband specifications, e.g. Butterworth filters.* □

While some results are given for the variance of this estimator, a similar technique described by Quinn and Fernandes [1991] allows easier derivation of a central limit theorem for the estimator.

3.1.2 The Technique of Quinn and Fernandes

Quinn and Fernandes [1991] have suggested estimating the parameters α and β in the following equation:

$$y_t - \beta y_{t-1} + y_{t-2} = \varepsilon_t - \alpha\varepsilon_{t-1} + \varepsilon_{t-2}.$$

subject to $\alpha = \beta$. The algorithm proceeds as follows (from Quinn and Fernandes [1991]):

1. Set $\alpha = \alpha_1 = 2\cos(\hat{\omega}_{\mathrm{i}})$ where $\hat{\omega}_{\mathrm{i}}$ is some initial estimate of ω_0 and set $j = 1$.

2. Filter the data to produce $\zeta_{t,j}$

$$\zeta_{t,j} = y_t + \alpha_j \zeta_{t-1,j} - \zeta_{t-2,j}; \quad t = 0, \ldots, T-1$$

   where $\zeta_{t,j} = 0$ for $t < 0$.

3. Form β_j by regressing $(\zeta_{t,j} + \zeta_{t-2,j})$ on $\zeta_{t-1,j}$

$$\beta_j = \frac{\sum_{t=0}^{T-1}(\zeta_{t,j} + \zeta_{t-2,j})\zeta_{t-1,j}}{\sum_{t=0}^{T-1}\zeta_{t-1,j}^2}$$

4. If $|\alpha_j - \beta_j|$ is small enough, set $\hat{\omega}_0 = \cos^{-1}(\frac{1}{2}\beta_j)$ and terminate. Otherwise, put $\alpha_{j+1} = \beta_j$, increment j and go to step 2.

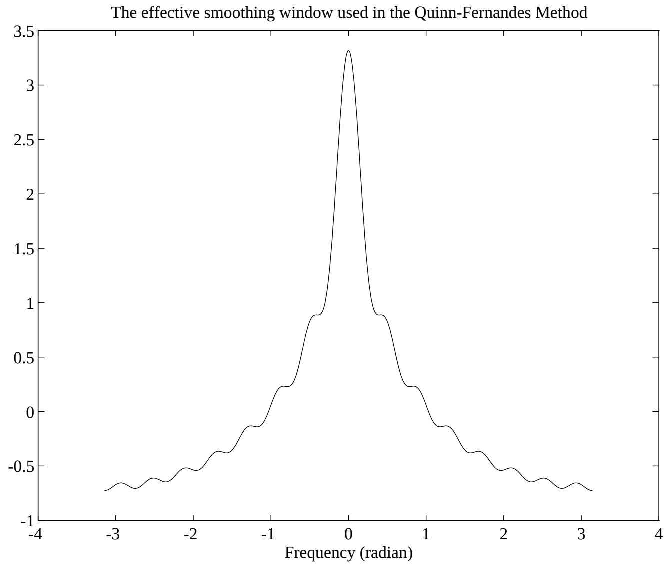

Figure 1: Plot of $\mu_T(\omega)$ as a function of frequency.

The algorithm may be interpreted (Quinn, Hannan and Huang [1993]) as finding a local maximum of the "smoothed periodogram" nearest $\hat{\omega}_{\mathrm{i}}$. The function maximised is

$$\kappa(\omega) = \int_{-\pi}^{\pi} I_y(\lambda)\mu_T(\omega - \lambda)d\lambda$$

where

$$\mu_T(\omega) = \sum_{k=1}^{T-1} k^{-1}\cos(k\omega).$$

An example of $\mu_T(\omega)$ for $T = 16$ is plotted in Figure 1.

Remark 16: *The maximiser of κ has the same central limit theorem (Theorem 3 of Quinn and Fernandes [1991]) as the periodogram maximiser:*

$$T^{3/2}(\hat{\omega}_0 - \omega_0) \sim N(0, 48\pi\rho^{-2}f_\varepsilon(\omega_0))$$

where $f_\varepsilon(\omega_0)$ is the spectral density of the noise at the true frequency. □

Remark 17: *The number of iterations required for convergence is small if a good initial estimate is used. For example, if $\hat{\omega}_{\mathrm{i}}$ is the discrete frequency periodogram maximiser, then only 2 or 3 iterations are generally sufficient.* □

Remark 18: *Due to the smoothing involved, the algorithm is robust to poor initial estimates $\hat{\omega}_{\mathrm{i}}$. Quinn and Fernandes [1991] show via simulation that even using Pisarenko's frequency estimate for $\hat{\omega}_{\mathrm{i}}$ is enough for convergence within 2 to 5 iterations.* □

3.2 On-line Filtering Techniques

Filtering techniques are of interest because it may be possible to modify them to be on-line or adaptive. When such techniques are adaptive, it is also possible to use them for frequency tracking.

3.2.1 The Hannan-Huang Estimator

Hannan and Huang [1993] noticed that an on-line version of the Quinn-Fernandes estimator may be formulated as follows.

Select a threshold parameter ϵ. Initialise, with $j = 1$,

$$t_j = 0 \; ; \; \alpha = \hat{\omega}_{\mathrm{i}}$$

For t from 0 to $T - 1$:

1. Set

$$\begin{aligned} \zeta_t &= e^{i\alpha}\zeta_{t-1} + y_t \; ; \; \zeta_{-1} = 0 \\ \hat{\omega}_t &= \arg\left[\sum_{k=t_j+1}^{t} (\zeta_k + y_k)\bar{\zeta}_{k-1}\right] \end{aligned} \tag{11}$$

2. Form

$$F_t(t_j) = \frac{1}{(t - t_j)^2}|\zeta_t|^2$$

3. If

$$F_t(t_j) < \epsilon \max_{t_j \le k < t} F_k(t_j)$$

   then set $j = j + 1$, $t_j = t$, $\alpha = \hat{\omega}_t$ and $\zeta_t = 0$.

4. Goto step 1.

Remark 19: *The reinitiation step is included because for a given value of $\Delta = \alpha - \omega_0$, $t - t_j$ (the time from the last reinitiation) must not be too large.* □

Remark 20: *From a practical view point, the reinitiation of the algorithm provided by step 3 is needed because the instability of the state update equation (11) means that $|\zeta_t|$ grows approximately linearly in t.* □

Remark 21: *Huang and Hannan [1993] have reformulated their approach to deal with signals modelled with time-varying frequency as in (5).* □

3.2.2 The Technique of Nehorai and Porat

The technique of Nehorai and Porat [1986] (also presented by Nehorai and Porat [1985]) is a recursive prediction error (Ljung and Söderström [1983]) approach for multi-harmonic frequency estimation. A simpler version modified for the single tone case is as follows.

We are given a noisy sinusoidal sequence, $\{y_t\}$ with $y_t = 0$ for $t < 0$ and $t > T - 1$ and an initial frequency estimate $\hat{\omega}_{\mathrm{i}}$. When $t \le 0$, set

$$\begin{array}{rcl} \hat{\omega}_t = \hat{\omega}_{\mathrm{i}} \; ; & \hat{a}_t = -2\cos(\hat{\omega}_t) & ; \; y'_t = y_t \\ \phi_t = 0 \; ; & \phi'_t = 0 & ; \; \psi_t = 0 \\ \epsilon_t = y_t \; ; & \epsilon'_t = y_t & ; \; \epsilon''_t = y_t \end{array}$$

and select dynamics for the variables ρ_t, r_t, γ_t and λ_t. For t=1 to $T-1$ perform the following recursions.

$$\begin{aligned}
\hat{\omega}_t &= \hat{\omega}_{t-1} + \frac{\gamma_{t-1}}{r_{t-1}}\psi_{t-1}\epsilon_{t-1} \\
\hat{a}_t &= -2\cos(\hat{\omega}_t) \\
y'_t &= y_t - \rho_t^2 y'_{t-2} - \rho_t \hat{a}_t y'_{t-1} \\
\phi_t &= \rho_t \epsilon'_{t-1} - y_{t-1} \\
\phi'_t &= \rho_t \epsilon''_{t-1} - y'_{t-1} \\
\psi_t &= 2\sin(\hat{\omega}_t)\phi'_t \\
\epsilon_t &= y_t + y_{t-2} - \rho_t^2 \epsilon'_{t-2} - \phi_t \hat{a}_{t-1} \\
\epsilon'_t &= y_t + y_{t-2} - \rho_t^2 \epsilon'_{t-2} - \phi_t \hat{a}_t \\
\epsilon''_t &= \epsilon'_t - \rho_t^2 \epsilon''_{t-2} - \rho_t \hat{a}_t \epsilon''_{t-1}
\end{aligned}$$

The dynamics of ρ_t, λ_t, γ_t and r_t are selected so as to satisfy convergence criteria set out by Ljung and Söderström [1983]. Nehorai and Porat [1986] selected the following dynamics.

$$\begin{aligned}
\rho_{t+1} &= \rho_0 \rho_t + (1-\rho_0)\rho_\infty \\
\lambda_{t+1} &= \lambda_0 \lambda_t + (1-\lambda_0) \\
\gamma_{t+1} &= \gamma_t / \left[\gamma_t + \lambda_{t+1}\right] \\
r_{t+1} &= r_t + \gamma_{t+1}\left[\psi_{t+1}^2 - r_t\right]
\end{aligned}$$

Remark 22: *The parameters ρ_0, ρ_∞ and λ_0 are user selectable.* □

The algorithm approximates ε_t by

$$\hat{\varepsilon}_t = h_t * y_t$$

where $*$ indicates convolution and h_t is (approximately) the impulse response of the filter

$$H(z) = \frac{1 - 2\cos(\hat{\omega}_t)z^{-1} + z^{-2}}{1 - 2\rho\cos(\hat{\omega}_t)z^{-1} + \rho^2 z^{-2}}.$$

where ρ is typically close to, but less than unity, to ensure the stability of $H(z)$. The parameter estimate $\hat{\omega}_t$ is updated so as to reduce

$$\sum_{t=0}^{T-1} \hat{\varepsilon}_t^2.$$

Remark 23: *Nehorai and Porat claim that their algorithm is asymptotically statistically efficient, however the only proof offered is the statement that their algorithm satisfies constraints provided by Ljung and Söderström [1983] which ensure the required result.* □

> Further Work 4: A thorough theoretical investigation of the performance of this algorithm for both frequency estimation and frequency tracking is needed.

Remark 24: *Since the Nehorai and Porat estimator is already adaptive, it is easily modified for the frequency tracking application by suitable choice of the tuning parameters.* □

4 Summary of Frequency Estimation Algorithms

Table 2 summarizes the various frequency estimators we have examined here.

4.1 Block Estimators

Of these estimators, the most attractive would appear to be the estimator of Quinn and Fernandes [1991], for several reasons. The estimator is unbiased, asymptotically efficient, requires fewer operations than full maximum likelihood and is more robust to initial conditions than that algorithm.

4.2 Fast Block Estimators

Of the weighted phase averaging estimators, that proposed by Lovell and Williamson [1992] has the best performance. The Kay [1989] estimator has similar performance for small noise levels, but its bias in the presence of unbounded, in particular Gaussian, noise is a problem.

4.3 On-Line Estimators

Interest in on-line estimators is because of the hope that they may be modified and applied to the frequency tracking problem. The Hannan-Huang estimator has been so modified (Huang and Hannan [1993]) and the Nehorai and Porat frequency estimator only requires a suitable choice of system dynamics to be used as a frequency tracker.

Frequency Estimator Summary			
Paradigm	Algorithm	Computational Complexity	Asymptotically Achieves Cramér-Rao Bound ?
ML and Approximate ML	Maximum Likelihood	$> O(T \log T)$	Yes
	Periodogram Maximiser	$> O(T \log T)$	Yes
	Discrete-Frequency Periodogram Maximiser	$O(T \log T)$	No
Fourier Coefficient	FTI1	$O(T \log T)$	No
	FTI2	$O(T \log T)$	No
	GPIE	$O(T \log T)$	No
Signal Subspace	Minimum Variance	$O(T^3)$	No
	Bartlett	$O(T^3)$	No
Noise Subspace	Pisarenko	$O(T)$	No
	MUSIC	$O(T^3)$	No
Phase Weighted Averaging	Lank-Reed-Pollon	$O(T)$	No
	Kay	$O(T)$	No
	Lovell	$O(T)$	Yes*
	Clarkson	$O(T)$	No
Filtering	Fernandes-Goodwin-de Souza	$O(T)$	Yes
	Quinn-Fernandes	$O(T)$	Yes
	Hannan-Huang	N/A	N/A
	Nehorai-Porat	N/A	N/A

* Further investigation of the asymptotic performance of this algorithm is needed.
N/A : Not applicable to on-line estimators.

Table 2: Summary of Frequency Estimators.

Part III

Frequency Tracking Algorithms

1 On-line Frequency Trackers

One reason for interest in on-line estimators of frequency is that some applications may not have access to blocked data and thus require continuously updating frequency estimates. Our major interest in these estimators is that they may lead to algorithms which tackle the the problem of frequency *tracking* or FM demodulation.

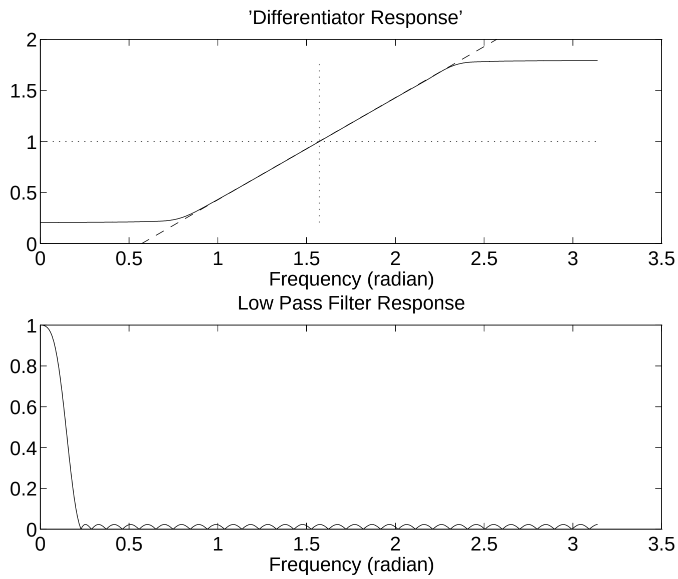

Figure 2: Filter magnitude responses used in the simple approach.

1.1 A Simple Filtering Approach

A continuous-time, analytic, noiseless, frequency modulated signal may be converted to an *amplitude* modulated waveform by differentiation:

$$\begin{aligned} z(t) &= \mu' + \rho \exp\left(i(\int_0^t \omega(t) dt + \phi) \right) \\ \frac{d}{dt} z(t) &= \rho \omega(t) \exp\left(i(\int_0^t \omega(t) dt + \phi) \right). \end{aligned}$$

An envelope detector can then be applied to extract $\omega(t)$ from $\frac{d}{dt} z(t)$.

Similarly, in discrete-time, the differentiation operation may be approximated by passing the complex signal

$$z_t = \mu' + \rho \exp \left(i(\omega_c t + \sum_{k=0}^{t} \lambda_k + \phi) \right),$$

with $\omega_c \gg \lambda_k$, through a linear-phase finite impulse response filter with magnitude response illustrated in the top diagram of Figure 2.

In the Figure, it is assumed that $\omega_c = \pi/2$. The dotted lines in the top diagram indicate that the filter gain at ω_c is unity, and the dashed line indicates that the slope of the magnitude response at ω_c is also unity.

The modulus of the filtered signal is then taken, and the result passed through a low pass filter with magnitude response given in the bottom diagram of Figure 2.

1.2 The Extended Kalman Filter

As noted in Remark 2, frequency tracking problems are state estimation (or prediction) problems — which are routinely solved using Kalman and extended Kalman filters.

With any application of Kalman filtering techniques, a signal model must first be specified. For example, the following signal model:

$$\begin{aligned}
\underline{x}_0 &= \begin{bmatrix} \omega \\ \phi \end{bmatrix} \\
\underline{x}_{t+1} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \underline{x}_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_t \\
&= F\underline{x}_t + G v_t \\
y_t &= \sin\left(\begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}_t \right) + \varepsilon_t \qquad (12) \\
&= \sin\left(\underline{l}' \underline{x}_t \right) + \varepsilon_t
\end{aligned}$$

with noise variance $\mathrm{E}[\varepsilon_t \varepsilon_s] = \sigma_\varepsilon^2 \delta_{t-s}$ and frequency variance $\mathrm{E}[v_t v_s] = \sigma_v^2 \delta_{t-s}$

Since the output equation (12) is non-linear, the extended Kalman filter equations

$$\begin{aligned}
\hat{\underline{x}}_{t|t} &= \hat{\underline{x}}_{t|t-1} + L_t \left[y_t - \sin\left(\underline{l}' \hat{\underline{x}}_{t|t-1} \right) \right] \\
\hat{\underline{x}}_{t+1|t} &= F \hat{\underline{x}}_{t|t} \\
L_k &= \Sigma_{k|k-1} H_k \Omega_k^{-1} \\
\Omega_k &= H_k' \Sigma_{k|k-1} H_k + \sigma_\varepsilon^2 \\
\Sigma_{k|k} &= \Sigma_{k|k-1} - \Sigma_{k|k-1} H_k \Omega_k^{-1} H_k' \Sigma_{k|k-1} \\
\Sigma_{k+1|k} &= F \Sigma_{k|k} F' + \sigma_v^2 G G' \\
H_k &= \underline{l} \cos\left(\underline{l}' \hat{\underline{x}}_{t|t-1} \right)
\end{aligned}$$

must be used (see Anderson and Moore [1979, page 195]).

Anderson and Moore [1979, pages 200-204] assume a sampled continuous-time signal rather than using a wholly discrete model.

Remark 25: *James [1992], Anderson, James and Williamson [1992] and James, Anderson and Williamson [1991a] have examined augmenting this signal model to take account of the the multi-harmonic frequency tracking and estimation problems.* □

Remark 26: *An aspect of the discrete-time frequency demodulation problem that these state-space approaches do not generally take into account is that the frequency estimate can only ever be in the range* $[0, 2\pi]$ *for the complex signal case and* $[0, \pi]$ *for the real signal case.* □

Anderson and Moore [1979] and Tam and Moore [1977] give some discussion on this point. Aspects of discrete-time frequency estimation are also discussed by Lovell, Kootsookos and Williamson [1991]. More general estimation problems on the circle are examined in the series of three papers by Lo and Willsky [1975c] and the two papers of Willsky [1974b].

For recursive discrete estimation, see the papers by Bitmead [1982], Bitmead and Anderson [1981] and Bitmead, Tsoi and Parker [1986].

1.3 The Gaussian 'Sum' Approach

The extended Kalman filter involves a linearisation of the problem around the current state, $\underline{x}_t$. Tam and Moore [1977] and Anderson and Moore [1979] have shown that an improvement in the demodulation performance of the extended Kalman filter may be obtained by using a bank of M such filters which are linearised about different points in the state-space. Each filter in the bank may have

- different initial states, $\underline{x}_{0,k}$,
- different frequency variances, $\sigma^2_{v,k}$ or
- different measurement noise variances, $\sigma^2_{\varepsilon,k}$.

After all filters are applied to the data several state estimates, ${\hat{\underline{x}}_{t|t}}^k$ are obtained. The final state estimate is given by

$$\hat{\underline{x}}_{t|t} = \sum_{k=1}^{M} \alpha_{t,k} {\hat{\underline{x}}_{t|t}}^k$$

where

$$\alpha_{t,k} = \frac{\alpha_{t-1,k}\gamma\left(y_t - \sin\left(\underline{l}' {\hat{\underline{x}}_{t|t-1}}^k\right), \Omega_{t,k}\right)}{\sum_{n=1}^{M} \alpha_{t-1,n}\gamma\left(y_t - \sin\left(\underline{l}' {\hat{\underline{x}}_{t|t-1}}^n\right), \Omega_{t,n}\right)}$$

and

$$\gamma(\mu, \Sigma) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)).$$

This approach involves approximating a non-Gaussian probability density function by a sum of Gaussian probability density functions. Such sums of Gaussian densities are usually referred to as Gaussian *mixtures* to avoid confusion with the addition of Gaussian random variables.

2 Hidden Markov Model Approaches

By imposing much structure on particular estimation problems, hidden Markov models (HMMs) have been used to good effect in the frequency tracking problem. Two approaches have been reported: frequency line tracking (Streit and Barrett [1990]) and a phase-frequency demodulation approach (White [1992]).

Other authors writing about these and similar methods include Barrett and Holdsworth [1992], Xie and Evans [1991] and Xie and Evans [1993].

> Further Work 5: One problem which needs to be addressed and is associated with the HMM approaches to frequency tracking is that of automatic model-parameter estimation. Automatic discretisation selection for the variable to be estimated when that variable is inherently continuous is another concern that needs to be examined. Other problems associated with HMMs are discussed by Anderson, James and Williamson [1992].

3 Summary of Frequency Tracking Algorithms

Frequency tracking procedures not covered in this report include the well-known phase locked loop (Kelly and Gupta [1972]; Polk and Gupta [1973]) and several of those discussed in Boashash [1992b]. The maximum likelihood polynomial phase method is one method described there which should be examined. Some authors see the phase locked loop as a simplification of the extended Kalman filter approach (James [1992]).

Estimators described by Boashash [1992b] based on time-frequency representations are extremely computationally intensive, and have been shown by Kootsookos, Lovell and Boashash [1992] to be arithmetically equivalent to weighted linear predictor techniques.

Remark 27: *One technique which may be worth investigation, but which has not been discussed here, is the probabilistic data association technique of Bar-Shalom and Fortmann [1988]. This method is of particular interest, because constraints such as track initiation (Colegrove [1992]) and track termination (Musicki, Evans and Stankovic [1992]) may be included in the algorithm formulation.* □

Part IV
Associated Problems

Along with the frequency estimation and tracking problems discussed in Part I , there are several associated problems which deserve discussion. These other problems include:

- Availability of Cramér-Rao lower bounds on the variance of unbiased parameter estimators and whether a particular estimator meets these bounds (is statistically efficient).
- Which performance indicators to use for the frequency estimation and tracking problems other than statistical efficiency (*e.g.* threshold onset).
- Whether to use the analytic signal or the real signal and, if the analytic signal should be used, how best to calculate it given only the real signal.
- Can the frequencies of two close-in-frequency tones be estimated accurately using frequency estimation algorithms. This is sometimes called the frequency resolution problem.
- For the multi-harmonic and multi-tone problems, the extra parameter which must be estimated is p, the model order. The robustness of multi-frequency algorithms to incorrect estimation of p should be examined.
- As indicated previously, when the frequency tracking problem is to be conducted in a high noise environment, the problems of track initiation and track termination become important.

We shall now examine some of these points.

1 Cramér-Rao Lower Bounds

The Cramér-Rao lower bound on the variance of an unbiased estimator of the frequency, $\hat{\omega}_0$ of a signal tone in noise is (see, for example, Rife and Boorstyn [1974])

$$\mathrm{var}(\hat{\omega}_0) \geq \frac{12\sigma^2}{T(T^2-1)\rho^2} \ . \tag{13}$$

For the multiharmonic frequency estimation problem, Barrett and McMahon [1987] have derived the analogous bound, which is

$$\mathrm{var}(\hat{\omega}_0) \geq \frac{12\sigma^2}{T(T^2-1)\sum_{k=1}^{p} k^2\rho_k^2} \ . \tag{14}$$

Remark 28: *Note that the effective signal energy is proportional to* $\sum_{k=1}^{p} k^2\rho_k^2$ *rather than* $\sum_{k=1}^{p} \rho_k^2$. □

Remark 29: *If an estimator is unbiased, and its variance approaches the Cramér-Rao lower bound as the sample size increases, then the estimator is said to be asymptotically efficient.* □

For the frequency tracking problem, asymptotic efficiency is generally not well defined, for obvious reasons. However, Peleg, Porat and Friedlander [1993] have derived Cramér-Rao bounds on the variance of instantaneous phase and instantaneous frequency estimators when the phase is known to be polynomial. The results are extended to the case of a continuous-phase signal in Peleg, Porat and Friedlander [1993].

2 Performance Indicators

2.1 Performance of Frequency Estimation Algorithms

Several performance indicators, other than asymptotic efficiency, may be of importance in the evaluation of frequency estimation algorithms. These include estimator bias, small sample size performance, computational complexity, and the thresholding performance of the estimators.

Estimator Bias Clearly, if we wish an estimator to be accurate then a major requirement should be that the estimator be asymptotically unbiased, *i.e.*

$$\mathrm{E}\left[\hat{\omega} - \omega_0\right] = 0$$

as the sample size increases ($T \to \infty$).

Unfortunately, it may not be obvious under what conditions a particular estimator is unbiased. For example, Quinn [1992a] has shown that Kay [1989] estimator is only unbiased when the amplitude of the noise is bounded — a condition which does not hold under the Gaussian noise assumption.

Small Sample Size Performance In some frequency estimation problems, only a small number of data points are available. In this case, asymptotic performance is not pertinent and any algorithm used must perform well on finite sample sizes.

Computational Complexity In real-time applications, the computational complexity of the algorithm used may be important.

Thresholding Analysis For some estimators it is possible to predict behaviour at all SNRs accurately.

This last point is particularly important as when the SNR is low, most estimators exhibit the *thresholding* phenomenon — which represents a marked decrease in performance for a relatively small change in SNR.

Further Work 6: A major study, incorporating these performance indicators, of the frequency estimators mentioned in Part II (and others) needs to be undertaken.

We give a brief account of various analyses which have been performed on some of the frequency estimation algorithms given in Part II.

2.1.1 Approximate Maximum Likelihood Frequency Estimator Performance

Rife and Boorstyn [1974] have analysed the performance of the frequency estimator based on the maximiser of the periodogram. Recently, Quinn and Kootsookos [1994] have obtained expressions which are simplified compared to those of Rife and Boorstyn. Figure 3 is a root mean square error versus SNR plot for various sample sizes obtained using the formulae presented in Quinn and Kootsookos [1994].

Another, different, approach is presented in James, Anderson and Williamson [1992a].

Remark 30: *Note that Rife and Boorstyn [1974] incorrectly state that they assume that the imaginary part of the additive noise is the Hilbert transform of the real part. If this were true, the real and imaginary parts of the noise would be correlated and the noise would therefore be coloured.* □

The analyses presented by both Rife and Boorstyn [1974] and Quinn and Kootsookos [1994] rely on the assumptions that

- the true frequency being a Fourier frequency (*i.e.* if the sample size is T then $\omega_0 = 2\pi k_0/T$) and
- the initial coarse frequency search is conducted over only the Fourier frequencies, $\omega_k = 2\pi k/T, \quad k = 0, 1, 2, \ldots, T-1$.

If either of these conditions is not met, then a decrease in the effective signal to noise ratio occurs.

> Further Work 7: The analysis of Quinn and Kootsookos [1994] should be extended to the real signal model and the approximate analytic signal cases.

Karan, Williamson and Anderson [1994] have analysed the case of model mismatch in the maximum likelihood frequency estimator, when the true signal has a frequency which is linearly increasing with time.

> Further Work 8: The work presented by Karan, Williamson and Anderson [1994] yields upper bounds on the penalty for assuming a constant frequency when in fact the signal has a linear frequency sweep. The bounds are not tight, and another approach, hopefully yielding tighter bounds, should be found.

James, Anderson and Williamson [1992b] and Williamson et al. [1994] have analysed the threshold performance of the multi-harmonic maximum likelihood frequency estimator. One of the main thrusts of Williamson et al. [1994] was to show that, for intermediate SNRs, rational harmonic outliers are the overwhelming mechanism by which the multi-harmonic maximum likelihood algorithm fails.

Figure 4 shows the theoretically derived and simulated results for the probability of A. rational harmonic outliers, B. noise-only outliers and C. any outlier. Comparison of A. and B. shows that, for intermediate SNRs, the rational harmonic outliers have a much higher probability of occurring than does a noise-only outlier.

> Further Work 9: Further work needs to be done to modify the algorithm to detect and reduce such rational harmonic outliers, and thereby improve the threshold-region performance of the algorithm.

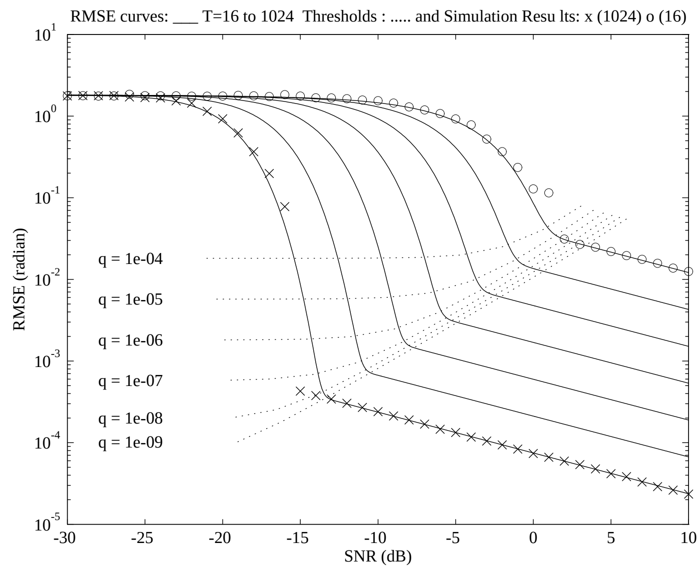

Figure 3: Performance curves for the Maximum Likelihood Estimator of frequency (complex signal case). The following curves are displayed:

— Theoretical Root Mean Square Error (RMSE) versus SNR for sample sizes $T = 16$ (topmost curve), 32, 64, 128, 256, 512 and 1024 (bottom curve).

o Sample RMSE versus SNR for sample size $T = 16$. Number of realisations per sample point is 2000.

x Sample RMSE versus SNR for sample size $T = 1024$. Number of realisations per sample point is 2000.

... constant probability of an outlier curves for $q = 1 \times 10^{-4}$ to $q = 1 \times 10^{-9}$

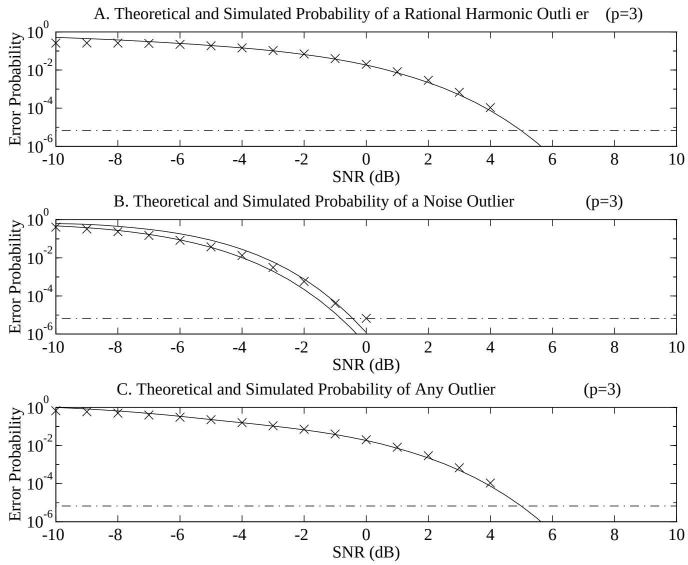

Figure 4: Outlier probability curves for the multiharmonic maximum likelihood estimator. The following curves are displayed:

A. (—) The probabilty of a rational harmonic outlier versus SNR calcualted theorectically.

A. (x) The probabilty of a rational harmonic outlier versus SNR obtained via simulation.

B. (—) The probabilty of a noise outlier versus SNR obtained theoretically (upper and lower bounds).

B. (x) The probabilty of a noise outlier versus SNR obtained via simulation.

C. (—) The probabilty of any outlier versus SNR obtained theoretically.

C. (x) The probabilty of any outlier versus SNR obtained via simulation.

2.1.2 Weighted-Phase Frequency Estimator Performance

Clarkson, Kootsookos and Quinn [1994] have analysed the performance of a class of weighted-phase frequency estimators, and have shown that such estimators are not asymptotically statistically efficient. What can be shown about the variance of such estimators is that they approach the Cramér-Rao lower bound for *fixed* sample size and large signal to noise ratio.

The analysis presented by Clarkson, Kootsookos and Quinn [1994] does not account for signal dependent (and therefore noise dependent) windows as examined by Lovell and Williamson [1992]. The analysis of weighted-phase frequency estimators which use signal dependent window functions presented there indicates that the variance of the estimators approaches the Cramér-Rao lower bound even for moderate signal to noise ratios.

The simulation studies presented by Lovell and Williamson [1992] suggest better performance of their estimator than those of Kay [1989].

3 The Analytic Signal

Many algorithms for frequency estimation, particularly the weighted phase-averaging techniques, rely upon the signal under analysis being analytic (Ville [1948]). Given that real-world data is not analytic (complex-valued), there is the issue of how the imaginary part of an analytic signal is generated and, consequently, how this problem affects statistical theory based on the analytic signal assumption.

Remark 31: *Very few results are available on the statistical effect that estimation of the quadrature signal has on frequency estimation procedures which assume it is available.* □

Figure 5 displays some simulation results where this effect is examined. The three frequency modulated signals

$$
\begin{aligned}
y_t^{\mathrm{r}} &= \sin\left(\omega_0 t + \phi\right) + \varepsilon_t \\
y_t^{\mathrm{h}} &= y_t^{\mathrm{r}} + i\hat{\mathrm{H}}\left[y_t^{\mathrm{r}}\right] \\
y_t^{\mathrm{c}} &= \exp\left(i(\omega_0 t + \phi)\right) + \varepsilon_t'
\end{aligned}
$$

were produced, with ε_t, the real part and imaginary part of ε_t' independent, identically distributed white Gaussian noise sequences and $\hat{\mathrm{H}}[y]$ indicating formation of an approximation to the Hilbert transform of y. The appropriate extended Kalman filter formulations of Anderson and Moore [1979, pp. 200–202] were then used to demodulate the two.

Remark 32: *The Figure indicates the improved performance of algorithms which use both in-phase and quadrature components.* □

Remark 33: *The demodulation performance of the EKF for the two in-phase and quadrature-phase sampled signals (y_t^{h} and y_t^{c}) is almost identical. This is despite not taking into account in the EKF signal model the noise colouration of y_t^{h}.* □

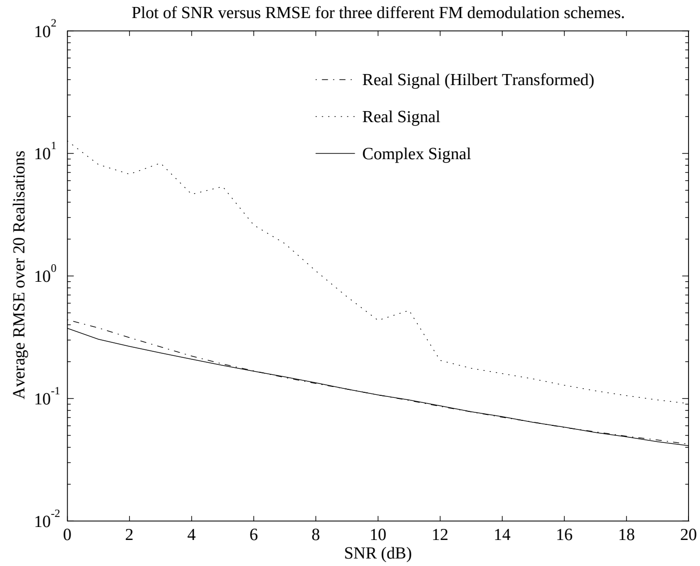

Figure 5: Plot of average frequency root mean-square error versus SNR for three different FM demodulation problems.

Further Work 10: An analysis of the effects of using

- the ideal analytic equivalent signal and
- block and on-line estimates of the analytic equivalent signal

in conjunction with the various frequency estimation and tracking algorithms needs to be conducted.

The authors Rihaczek [1966], Bedrosian [1963] and Nuttall [1966] have proved various results about narrow-band frequency modulated signals and their Hilbert transforms.

4 Frequency Resolution

When the signal of interest is two closely spaced sinusoids

$$y_t = \mu + \rho \cos\left(\omega_0(t-\nu)+\phi\right) + \rho\cos\left((\omega_0 + \frac{a}{T})(t-\nu)+\phi\right) + \varepsilon_t \tag{15}$$

then Hannan and Quinn [1989] have shown that one-dimensional periodogram techniques are outperformed by the maximiser of the regression sum of squares. The regression sum of squares for the signal model of (15) will be a function of two variables, ω_0 and a.

Other approaches are discussed by van Hamme [1991] and Lee [1992].

5 Model Order Selection

In any model-based scheme, the issue of appropriate model complexity or model order arises. In the multi-tone and multi-harmonic frequency estimation problems, this is the selection of p, the model order. If p is too large, then spurious frequencies may appear to be present; if p is too small, then important information about some frequencies may be lost.

The standard method for model order selection in auto-regressive modelling is the automatic information criterion or AIC proposed by Akaike [1974]. Quinn [1989] and Hannan [1992] have suggested modifications to this criterion.

> Further Work 11: Further study of both model order selection and the robustness of particular estimators to inaccurate order estimates is required. Some work on this last point, for the multiharmonic case, has been reported in Williamson et al. [1994].

6 Summary of the Associated Problems

This ends the section on problems associated with frequency estimation and tracking. The main problems identified here which need further work are the effect of using the analytic signal for either problem and a full theoretical and experimental comparison of the suite of frequency estimators available.

The next two parts of this report detail some frequency estimation (Part II) and frequency tracking (Part III) algorithms.

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