

An Introduction to Mathematical Chaos Theory and Fractal Geometry



For a printable version of this document, please click [here](#).

The following essay was compiled by me, Manus J. Donahue III (second year Physics and Philosophy major at Duke University..age 19). It has been unofficially published in four different countries, has been cited in The New York Times and has been awarded technology site of the day by TechSightings.com. Please cite this page as a reference if you use any of the material on this page in essays, documents, or presentations. Also, you may e-mail me at mjd@duke.edu if you have any questions, and I'll try to get back with you as soon as possible.

Because I compiled this essay for myself and the enjoyment of others, and because I am presenting it completely free, I am not responsible for any copyright violations or anything like that. Some of the pictures that are included in this essay (although almost universally common) were taken from other Web pages. If you are a high school/middle school student who has to do a report on chaos theory and you print this essay off and turn it in, you will be violating not only the work of myself, but the various other people who unknowingly may have contributed to this site. Don't do that - these people deserve credit for their work! Use this paper merely as a "jumping off point" for your own research, and then write a paper that is even better – and publish it. I wrote this essay because I was always fascinated by chaos theory and non-linear math and I could never find explanative essays aimed at the "average person." Hopefully this one is.

Finally, I get a lot of e-mails from people asking me to recommend books and other resources on chaos theory. The books that I used while writing this paper are included below; they're the ones that I would recommend:

Chaos: Making a New Science by James Gleick

Fractals: The Patterns of Chaos by John Briggs

In the Wake of Chaos by Stephen H. Kellert

Needless to say, there is also a surfeit of resources on the Web concerning chaos theory and fractals – so they would be worth checking out too.

"Physicists like to think that all you have to do is say, these are the conditions, now

what happens next?" -Richard P. Feynman

The world of mathematics has been confined to the linear world for centuries. That is to say, mathematicians and physicists have overlooked dynamical systems as random and unpredictable. The only systems that could be understood in the past were those that were believed to be linear, that is to say, systems that follow predictable patterns and arrangements. Linear equations, linear functions, linear algebra, linear programming, and linear accelerators are all areas that have been understood and mastered by the human race. However, the problem arises that we humans do not live in an even remotely linear world; in fact, our world should indeed be categorized as nonlinear; hence, proportion and linearity is scarce. How may one go about pursuing and understanding a nonlinear system in a world that is confined to the easy, logical linearity of everything? This is the question that scientists and mathematicians became burdened with in the 19th Century; hence, a new science and mathematics was derived: chaos theory.

The very name "chaos theory" seems to contradict reason, in fact it seems somewhat of an oxymoron. The name "chaos theory" leads the reader to believe that mathematicians have discovered some new and definitive knowledge about utterly random and incomprehensible phenomena; however, this is not entirely the case. The acceptable definition of chaos theory states, chaos theory is the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems. A dynamical system may be defined to be a simplified model for the time-varying behavior of an actual system, and aperiodic behavior is simply the behavior that occurs when no variable describing the state of the system undergoes a regular repetition of values. Aperiodic behavior never repeats and it continues to manifest the effects of any small perturbation; hence, any prediction of a future state in a given system that is aperiodic is impossible. Assessing the idea of aperiodic behavior to a relevant example, one may look at human history. History is indeed aperiodic since broad patterns in the rise and fall of civilizations may be sketched; however, no events ever repeat exactly. What is so incredible about chaos theory is that unstable aperiodic behavior can be found in mathematically simple systems. These very simple mathematical systems display behavior so complex and unpredictable that it is acceptable to merit their descriptions as random.

An interesting question arises from many skeptics concerning why chaos has just recently been noticed. If chaotic systems are so mandatory to our every day life, how come mathematicians have not studied chaos theory earlier? The answer can be given in one word: computers. The calculations involved in studying chaos are repetitive, boring and number in the millions. No human is stupid enough to endure the boredom; however, a computer is always up to the challenge. Computers have always been known for their excellence at mindless repetition; hence, the computer is our telescope when studying chaos. For, without a doubt, one cannot really explore chaos without a computer.

Before advancing into the more precocious and advanced areas of chaos, it is necessary to touch on the basic principle that adequately describes chaos theory, the Butterfly Effect. The Butterfly Effect was vaguely understood centuries ago and is still satisfactorily portrayed in folklore:

"For want of a nail, the shoe was lost;
 For want of a shoe, the horse was lost;
 For want of a horse, the rider was lost;
 For want of a rider, the battle was lost;
 For want of a battle, the kingdom was lost!"

Small variations in initial conditions result in huge, dynamic transformations in concluding events. That is to say that there was no nail, and, therefore, the kingdom was lost. The graphs of what seem to be identical, dynamic systems appear to diverge as time goes on until all resemblance disappears.

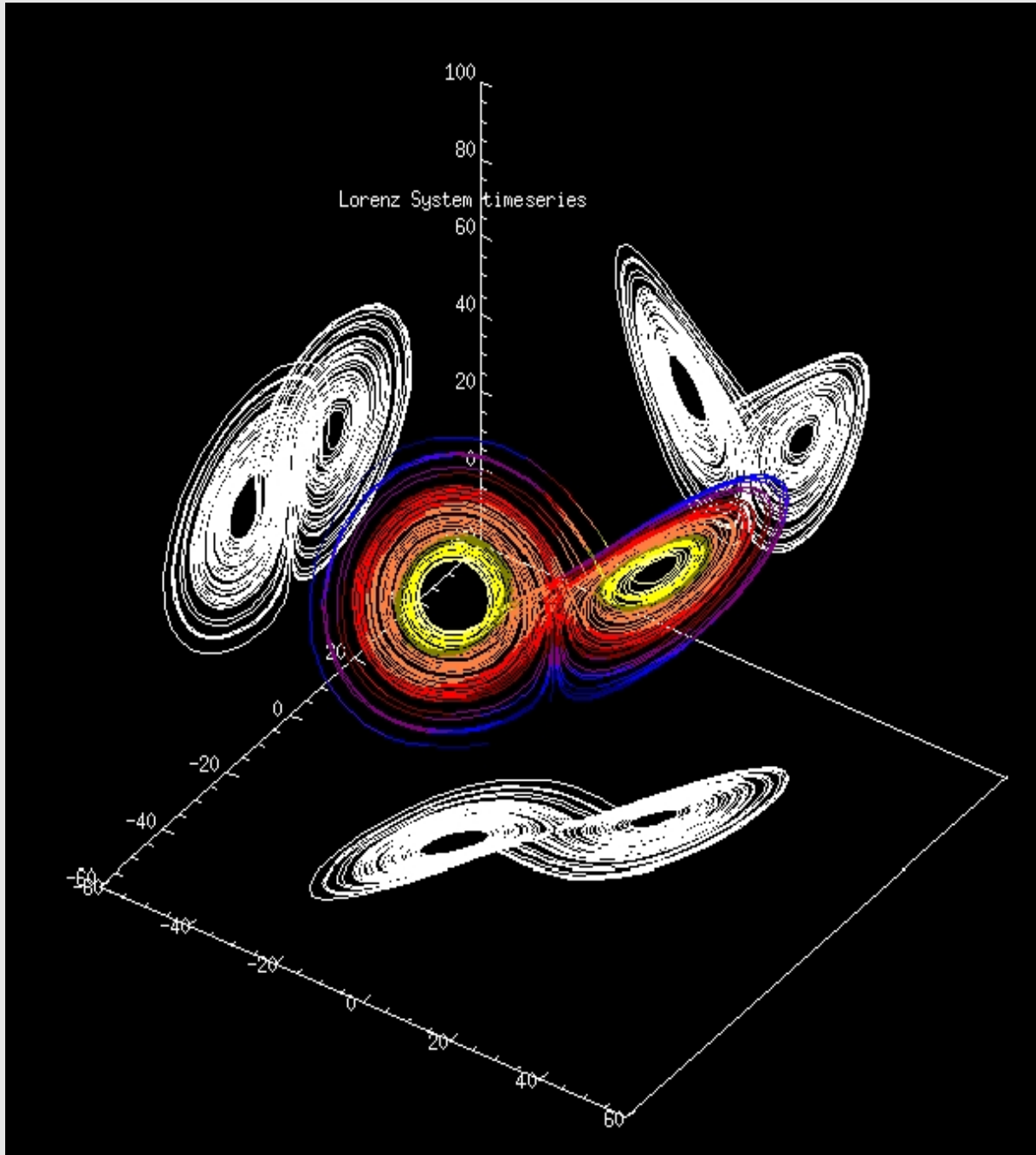
Perhaps the most identifiable symbol linked with the Butterfly Effect is the famed Lorenz Attractor. Edward Lorenz, a curious meteorologist, was looking for a way to model the action of the chaotic behavior of a gaseous system. Hence, he took a few equations from the physics field of fluid dynamics, simplified them, and got the following three-dimensional system:

$$\begin{aligned}dx/dt &= \Delta(y-x) \\ dy/dt &= r \cdot x - y - x \cdot z \\ dz/dt &= x \cdot y - b \cdot z\end{aligned}$$

Delta represents the "Prandtl number," the ratio of the fluid viscosity of a substance to its thermal conductivity; however, one does not have to know the exact value of this constant; hence, Lorenz simply used 10. The variable "r" represents the difference in temperature between the top and bottom of the gaseous system. The variable "b" is the width to height ratio of the box which is being used to hold the gas in the gaseous system. Lorenz used 8/3 for this variable. The resultant x of the equation represents the rate of rotation of the cylinder, "y" represents the difference in temperature at opposite sides of the cylinder, and the variable "z" represents the deviation of the system from a linear, vertical graphed line representing temperature. If one were to plot the three differential equations on a three-dimensional plane, using the help of a computer of course, no geometric structure or even complex curve would appear; instead, a weaving object known as the Lorenz Attractor appears. Because the system never exactly repeats itself, the trajectory never intersects itself. Instead it loops around forever. I have included a computer animated Lorenz Attractor which is quite similar to the production of Lorenz himself. The following Lorenz Attractor was generated by running data through a 4th-order Runge-Kutta fixed-timestep integrator with a step of .0001, printing every 100th data point. It ran for 100 seconds, and only took the last 4096 points. The original parameters were $a=16$, $r=45$, and $b=4$ for the following equations (similar to the original Lorenz equations):

$$\begin{aligned}x' &= a(y-x) \\ y' &= rx - y - xz \\ z' &= xy - bz\end{aligned}$$

The initial position of the projectory was (8,8,14). When the points were generated and graphed, the Lorenz Attractor was produced in 3-D:



The attractor will continue weaving back and forth between the two wings, its motion seemingly random, its very action mirroring the chaos which drives the process. Lorenz had obviously made an immense breakthrough in not only chaos theory, but life. Lorenz had proved that complex, dynamical systems show order, but they never repeat. Since our world is classified as a dynamical, complex system, our lives, our weather, and our experiences will never repeat; however, they should form patterns.

Lorenz, not quite convinced with his results, did a follow-up experiment in order to support his previous conclusions. Lorenz established an experiment that was quite simple; it is known today as the Lorenzian Waterwheel. Lorenz took a waterwheel; it had about eight

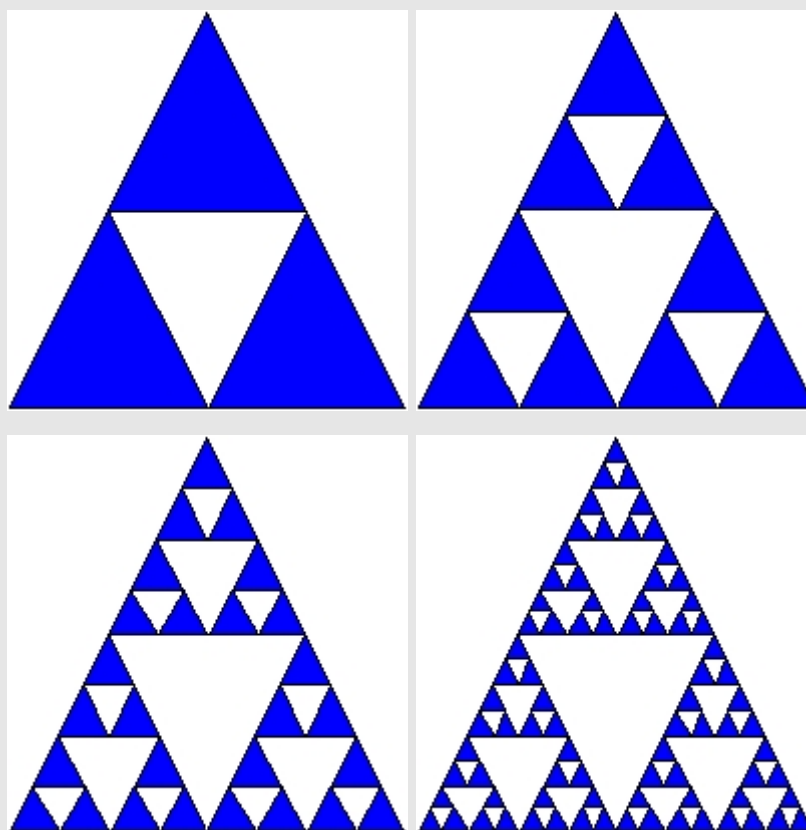
buckets spaced evenly around its rim with a small hole at the bottom of each . The buckets were mounted on swivels, similar to Ferris-wheel seats, so that the buckets would always point upwards. The entire system was placed under a waterspout. A slow, constant stream of water was propelled from the waterspout; hence, the waterwheel began to spin at a fairly constant rate. Lorenz decided to increase the flow of water, and, as predicted in his Lorenz Attractor, an interesting phenomena arose. The increased velocity of the water resulted in a chaotic motion for the waterwheel. The waterwheel would revolve in one direction as before, but then it would suddenly jerk about and revolve in the opposite direction. The filling and emptying of the buckets was no longer synchronized; the system was now chaotic. Lorenz observed his mysterious waterwheel for hours, and, no matter how long he recorded the positions and contents of the buckets, there was never an instance where the waterwheel was in the same position twice. The waterwheel would continue on in chaotic behavior without ever repeating any of its previous conditions. A graph of the waterwheel would resemble the Lorenz Attractor.

Now it may be accepted from Lorenz and his comrades that our world is indeed linked with an eerie form of chaos. Chaos and randomness are no longer ideas of a hypothetical world; they are quite realistic here in the status quo. A basis for chaos is established in the Butterfly Effect, the Lorenz Attractor, and the Lorenz Waterwheel; therefore, there must be an immense world of chaos beyond the rudimentary fundamentals. This new form mentioned is highly complex, repetitive, and replete with intrigue.

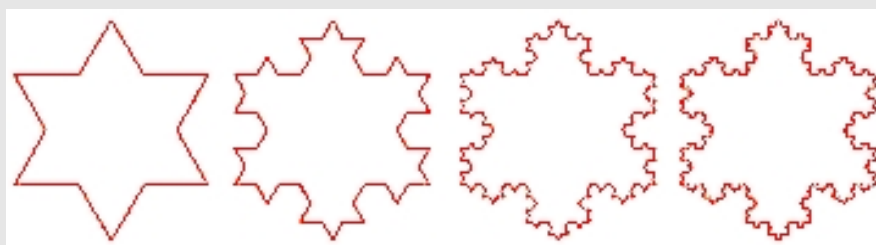
"I coined fractal from the Latin adjective fractus. The corresponding Latin verb frangere means "to break": to create irregular fragments. It is therefore sensible-and how appropriate for our needs!-that, in addition to "fragmented", fractus should also mean "irregular," both meanings being preserved in fragment." -Benoit Mandelbrot

The extending and folding of chaotic systems give strange attractors, such as the Lorenz Attractor, the distinguishing characteristic of a nonintegral dimension. This nonintegral dimension is most commonly referred to as a fractal dimension. Fractals appear to be more popular in the status quo for their aesthetic nature than they are for their mathematics. Everyone who has seen a fractal has admired the beauty of a colorful, fascinating image, but what is the formula that makes up this glitzy image? The classical Euclidean geometry that one learns in school is quite different than the fractal geometry mainly because fractal geometry concerns nonlinear, nonintegral systems while Euclidean geometry is mainly oriented around linear, integral systems. Hence, Euclidean geometry is a description of lines, ellipses, circles, etc. However, fractal geometry is a description of algorithms. There are two basic properties that constitute a fractal. First, is self-similarity, which is to say that most magnified images of fractals are essentially indistinguishable from the unmagnified version. A fractal shape will look almost, or even exactly, the same no matter what size it is viewed at. This repetitive pattern gives fractals their aesthetic nature. Second, as mentioned earlier, fractals have non-integer dimensions. This means that they are entirely different from the graphs of lines and conic sections that we have learned about in fundamental Euclidean geometry classes. By taking the midpoints of each side of an equilateral triangle and connecting them together, one gets an interesting fractal known as the Sierpinski Triangle. The iterations are repeated an infinite number of times and eventually a very

simple fractal arises:



In addition to the famous Sierpinski Triangle, the Koch Snowflake is also a well noted, simple fractal image. To construct a Koch Snowflake, begin with a triangle with sides of length 1. At the middle of each side, add a new triangle one-third the size; and repeat this process for an infinite amount of iterations. The length of the boundary is $3 \times \frac{4}{3} \times \frac{4}{3} \times \frac{4}{3} \dots$ -infinity. However, the area remains less than the area of a circle drawn around the original triangle. What this means is that an infinitely long line surrounds a finite area. The end construction of a Koch Snowflake resembles the coastline of a shore.



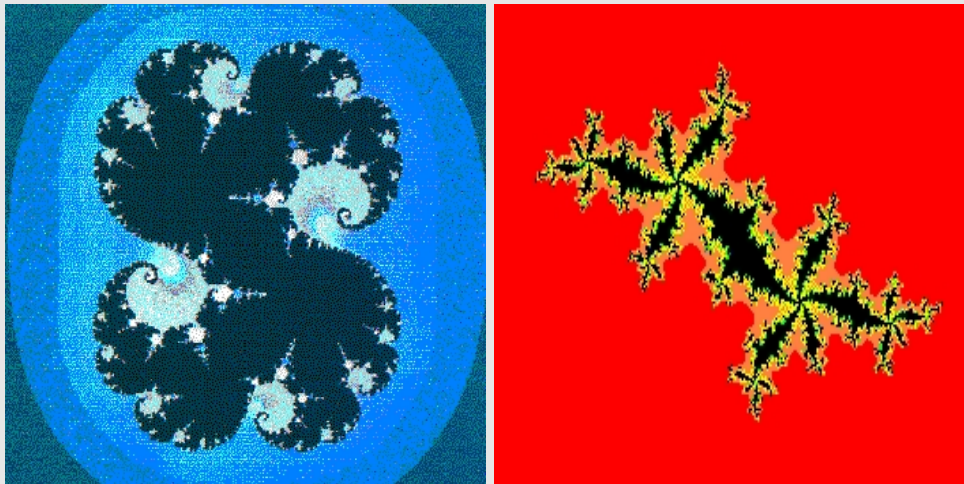
The two fundamental fractals that I have included provided a basis for much more complex, elaborate fractals. Two of the leading researchers in the field of fractals were Gaston Maurice Julia and Benoit Mandelbrot. Their discoveries and breakthroughs will be discussed next.

On February 3rd, 1893, Gaston Maurice Julia was born in Sidi Bel Abbes, Algeria. Julia was injured while fighting in World War I and was forced to wear a leather strap across his

face for the rest of his life in order to protect and cover his injury. he spent a large majority of his life in hospitals; therefore, a lot of his mathematical research took place in the hospital. At the age of 25, Julia published a 199 page masterpiece entitled "Memoire sur l'iteration des fonctions." The paper dealt with the iteration of a rational function. With the publication of this paper came his claim to fame. Julia spent his life studying the iteration of polynomials and rational functions. If $f(x)$ is a function, various behaviors arise when "f" is iterated or repeated. If one were to start with a particular value for x , say $x=a$, then the following would result:

$$a, f(a), f(f(a)), f(f(f(a))), \text{ etc.}$$

Repeatedly applying "f" to "a" yields arbitrarily large values. Hence, the set of numbers is partitioned into two parts, and the Julia set associated to "f" is the boundary between the two sets. The filled Julia set includes those numbers $x=a$ for which the iterates of "f" applied to "a" remain bounded. The following fractals belong to the Julia set.



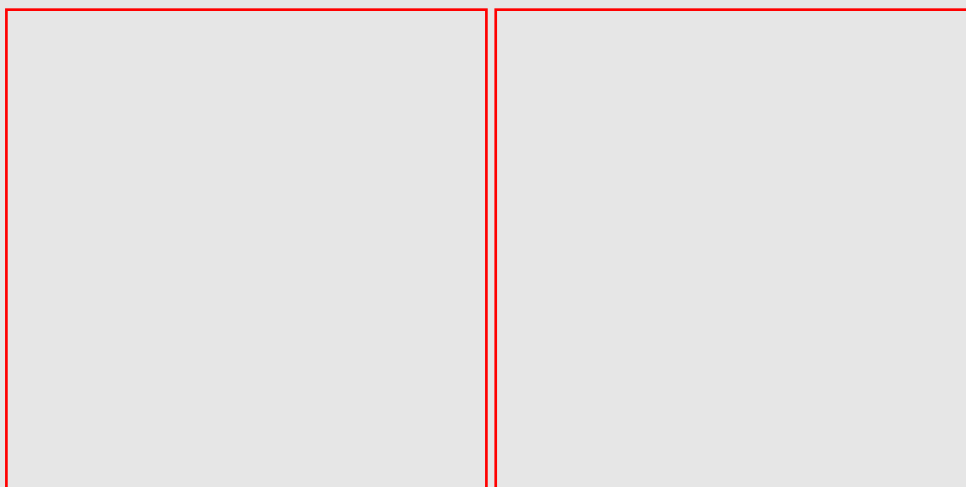
Julia became famous around the 1920's; however, upon his demise, he was essentially forgotten. It was not until 1970 that the work of Gaston Maurice Julia was revived and popularized by Polish born Benoit Mandelbrot.

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." -Benoit Mandelbrot

Benoit Mandelbrot was born in Poland in 1924. When he was 12 his family emigrated to France and his uncle, Szolem Mandelbrot, took responsibility for his education. It is said that Mandelbrot was not very successful in his schooling; in fact, he may have never learned his multiplication tables. When Benoit was 21, his uncle showed him Julia's important 1918 paper concerning fractals. Benoit was not overly impressed with Julia's work, and it was not until 1977 that Benoit became interested in Julia's discoveries. Eventually, with the aid of computer graphics, Mandelbrot was able to show how Julia's work was a source of some of the most beautiful fractals known today. The Mandelbrot set is made up of connected points in the complex plane. The simple equation that is the basis of the Mandelbrot set is included below.

$$\text{changing number} + \text{fixed number} = \text{Result}$$

In order to calculate points for a Mandelbrot fractal, start with one of the numbers on the complex plane and put its value in the "Fixed Number" slot of the equation. In the "Changing number" slot, start with zero. Next, calculate the equation. Take the number obtained as the result and plug it into the "Changing number" slot. Now, repeat (iterate) this operation an infinite number of times. When iterative equations are applied to points in a certain region of the complex plane, a fractal from the Mandelbrot set results. A few fractals from the Mandelbrot set are included below.



Benoit Mandelbrot currently works at IBM's Watson Research Center. In addition, he is a Professor of the Practice of Mathematics at Harvard University. He has been awarded the Barnard Medal for Meritorious Service to Science, the Franklin Medal, the Alexander von Humboldt Prize, the Nevada Medal, and the Steinmetz Medal. His work with fractals has truly influenced our world immensely.

It is now established that fractals are quite real and incredible; however, what do these newly discovered objects have to do with real life? Is there a purpose behind these fascinating images? The answer is a somewhat surprising yes. Homer Smith, a computer engineer of Art Matrix, once said, "If you like fractals, it is because you are made of them. If you can't stand fractals, it's because you can't stand yourself." Fractals make up a large part of the biological world. Clouds, arteries, veins, nerves, parotid gland ducts, and the bronchial tree all show some type of fractal organization. In addition, fractals can be found in regional distribution of pulmonary blood flow, pulmonary alveolar structure, regional myocardial blood flow heterogeneity, surfaces of proteins, mammographic parenchymal pattern as a risk for breast cancer, and in the distribution of arthropod body lengths. Understanding and mastering the concepts that govern fractals will undoubtedly lead to breakthroughs in the area of biological understanding. Fractals are one of the most interesting branches of chaos theory, and they are beginning to become ever more key in the world of biology and medicine.

George Cantor, a nineteenth century mathematician, became fascinated by the infinite

number of points on a line segment. Cantor began to wonder what would happen when an infinite number of line segments were removed from an initial line interval. Cantor devised an example which portrayed classical fractals made by iteratively taking away something. His operation created a "dust" of points; hence, the name Cantor Dust. In order to understand Cantor Dust, start with a line; remove the middle third; then remove the middle third of the remaining segments; and so on. The operation is shown below.



The Cantor set is simply the dust of points that remain. The number of these points are infinite, but their total length is zero. Mandelbrot saw the Cantor set as a model for the occurrence of errors in an electronic transmission line. Engineers saw periods of errorless transmission, mixed with periods when errors would come in gusts. When these gusts of errors were analyzed, it was determined that they contained error-free periods within them. As the transmissions were analyzed to smaller and smaller degrees, it was determined that such dusts, as in the Cantor Dust, were indispensable in modeling intermittency.

"It's an experience like no other experience I can describe, the best thing that can happen to a scientist, realizing that something that's happened in his or her mind exactly corresponds to something that happens in nature. It's startling every time it occurs. One is surprised that a construct of one's own mind can actually be realized in the honest-to-goodness world out there. A great shock, and a great, great joy." -Leo Kadanoff

The fractals and iterations are fun to look at; the Cantor Dust and Koch Snowflakes are fun to think about, but what breakthroughs can be made in terms of discovery? Is chaos theory anything more than a new way of thinking? The future of chaos theory is unpredictable, but if a breakthrough is made, it will be huge. However, miniature discoveries have been made in the field of chaos within the past century or so, and, as expected, they are mind boggling.

The first consumer product to exploit chaos theory was produced in 1993 by Goldstar Co. in the form of a revolutionary washing machine. A chaotic washing machine? The washing machine is based on the principle that there are identifiable and predictable movements in nonlinear systems. The new washing machine was designed to produce cleaner and less tangled clothes. The key to the chaotic cleaning process can be found in a small pulsator that rises and falls randomly as the main pulsator rotates. The new machine was surprisingly successful. However, Daewoo, a competitor of Goldstar claims that they first started commercializing chaos theory in their "bubble machine" which was released in 1990. The "bubble machine" was the first to use the revolutionary "fuzzy logic circuits." These circuits are capable of making choices between zero and one, and between true and

false. Hence, the "fuzzy logic circuits" are responsible for controlling the amount of bubbles, the turbulence of the machine, and even the wobble of the machine. Indeed, chaos theory is very much a factor in today's consumer world market.

The stock markets are said to be nonlinear, dynamic systems. Chaos theory is the mathematics of studying such nonlinear, dynamic systems. Does this mean that chaoticians can predict when stocks will rise and fall? Not quite; however, chaoticians have determined that the market prices are highly random, but with a trend. The stock market is accepted as a self-similar system in the sense that the individual parts are related to the whole. Another self-similar system in the area of mathematics are fractals. Could the stock market be associated with a fractal? Why not? In the market price action, if one looks at the market monthly, weekly, daily, and intra day bar charts, the structure has a similar appearance. However, just like a fractal, the stock market has sensitive dependence on initial conditions. This factor is what makes dynamic market systems so difficult to predict. Because we cannot accurately describe the current situation with the detail necessary, we cannot accurately predict the state of the system at a future time. Stock market success can be predicted by chaoticians. Short-term investing, such as intra day exchanges are a waste of time. Short-term traders will fail over time due to nothing more than the cost of trading. However, over time, long-term price action is not random. Traders can succeed trading from daily or weekly charts if they follow the trends. A system can be random in the short-term and deterministic in the long term.

Perhaps even more important than stock market chaos and predictability is solar system chaos. Astronomers and cosmologists have known for quite some time that the solar system does not "run with the precision of a Swiss watch." Inabilities occur in the motions of Saturn's moon Hyperion, gaps in the asteroid belt between Mars and Jupiter, and in the orbit of the planets themselves. For centuries astronomers tried to compare the solar system to a gigantic clock around the sun; however, they found that their equations never actually predicted the real planets' movement. It is easy to understand how two bodies will revolve around a common center of gravity. However, what happens when a third, fourth, fifth or infinite number of gravitational attractions are introduced? The vectors become infinite and the system becomes chaotic. This prevents a definitive analytical solution to the equations of motion. Even with the advanced computers that we have today, the long term calculations are far too lengthy. Stephen Hawking once said, "If we find the answer to that (the universe), it would be the ultimate triumph of human reason-for then we would know the mind of God.

The applications of chaos theory are infinite; seemingly random systems produce patterns of spooky understandable irregularity. From the Mandelbrot set to turbulence to feedback and strange attractors; chaos appears to be everywhere. Breakthroughs have been made in the past in the area chaos theory, and, in order to achieve any more colossal accomplishments in the future, they must continue to be made. Understanding chaos is understanding life as we know it.

However, if we do discover a complete theory, it should in time be understandable in broad principle by everyone, not just a few scientists. Then we shall all, philosophers,

scientists, and just ordinary people, be able to take part in the discussion of the question of why it is that we and the universe exist.

-Stephen Hawking

TC