Bridging Level-K to Nash Equilibrium

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Abstract

We propose a new solution concept, NLK that connects Nash Equilibrium (NE) and Level-K. It allows a player in a game to believe that her opponent may be either less- or as sophisticated as, she is–a view with support in psychology. We apply it to data from four published papers on static, dynamic and auction games. NLK provides different predictions than those of NE and Level-K. Moreover, a simple version of it explains the experimental data better in many cases, with the same or lower number of parameters. We discuss extensions to games with more than two players and heterogeneous beliefs.

Key words: Nash Equilibrium, Level-K, Bounded Rationality

JEL: D01, C72, C92

1. Introduction

There is mounting and robust evidence from laboratory experiments of substantial discrepancies between the prediction of Nash Equilibrium (NE) and the behavior of agents.[2] Among all the alternative models that retain the individual rationality, but relax correct beliefs, Level-K is probably the most prominent one.[3] First proposed by Stahl and Wilson (1994, 1995) and Nagel (1995), Level-K introduces a non-equilibrium, structural model of strategic thinking, which admits possible cognitive limitations of players that are not allowed in NE.[4] This model has a hierarchy of levels of sophistication that are constructed iteratively starting with an exogenous,

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[2] There is much experimental evidence that predictions of both (Bayesian) NE in static games and Subgame Perfect Nash Equilibrium (SPNE) in dynamic games fail miserably. For instance, see McKelvey and Palfrey (1992) and Kagel and Levin (2002).

[3] Another strand of models such as Quantal Response Equilibrium (McKelvey and Palfrey 1995) retains correct beliefs but allow errors in best response.

[4] There are many variations and extensions of the Level-K model and we refer the reader to Crawford, Costa-Gomes, and Iriberri (2013) and the references therein.

non-strategic and least sophisticated $level_0$ player. Higher levels are then constructed by assuming that a $level_k$ player best responds to $level_{k-1}$ opponents, $k = 1,2, \ldots$. Absent in NE, the Level-K model explicitly allows players to consider their opponents as less sophisticated than themselves. However, it does not allow players to use an important element of strategic thinking, namely, "put yourself in the other's shoes."

Our paper introduces a new solution concept, NLK, that bridges between NE and the Level-K model. Whereas a Nash player believes that the other player is another Nash player, and a Level-K player believes that the other player is less sophisticated than herself, NLK allows the player to believe, with a probability λ, the other player can be a naïve player, less sophisticated, than herself and, with a probability of $(1-\lambda)$, another NLK player, as sophisticated as herself. However, NLK player is still best responding to her subjective beliefs like in both NE and Level-K.[5] We also discuss below how to construct a hierarchy of levels as Level-K does. However, in this work we compare the performance of NLK, employing only $k = 1$, to that Nash equilibrium or Level-K model with several k's, where the naïve player is consistently a random $level_0$ player that chooses uniformly among its strategy set.

Our model has two possible interpretations:

1. **A population game**: In this interpretation, an NLK player behaves as if she faces a population composed of naïve players and NLK players. In equilibrium, an NLK player best responds to her belief that with a probability λ, her opponent is a naïve player, and that with a probability of $(1-\lambda)$, her opponent is another NLK player (like herself). Note that λ is the subjective belief formed by the player and it does not have to coincide with the objective proportions of naïve players in the population, denoted by ρ. Thus, with $\lambda \neq \rho$, NLK is not a "full-equilibrium,"[6] as it allows an NLK player to hold inconsistent beliefs regarding the proportion of naïve players in the population. Such inconsistency finds support in psychology: The "False Consensus Effect," first introduced by Ross, Greene, and House (1977), claims that people overestimate the proportion of people like

[5] The formal definition of NLK and its extensions to Bayesian games and dynamic games are in Section 2.
[6] Stahl and Wilson (1995) include a rational expectation type together with different types of $level_k$ and Nash players in analyzing experimental data of a 3×3 symmetric game. Their results reject the existence of rational expectation type.

themselves($\lambda < \rho$).[7] More recent works, both in psychology and experimental economics, have re-evaluated the "False Consensus Effect" with some works providing evidence in support of such effect (Krueger and Clement 1994, Jimenez-Gomez 2018), while other works point at evidence to an opposite effect ($\lambda > \rho$) (Dawes 1990, Sherman, Presson, and Chassin 1984) or the absence of a biased belief (Engelmann and Strobel 2000) Of course, one may insist on consistency by requiring that in a "full-equilibrium" $\lambda = \rho$.

2. **A hierarchy of heterogeneous players**: A construction of such hierarchy can be accomplished in two ways.

a. As an analog of the Level-K model: A player is an NLK player of type m, denote by NLK_m, when his naïve opponent is exogenously given as a $level_{m-1}$ player of the Level-K model. Thus, an NLK_m player coincides with a $level_m$ player when $\lambda = 1$ and NLK equilibrium reduces to NE when $\lambda = 0$.

b. **As an analog of the Poisson Cognitive Hierarchy (P-CH) model** (Camerer, Ho, and Chong 2004).[8]

In this work we only use $m = 1$, resulting in NLK that has just one parameter, λ. We show that this simplest version of NLK already outperforms Level-K in many cases, although in some of them Level-K uses more than one parameter.

To illustrate the NLK equilibrium, consider a simple example of the chicken game introduced by Rapoport and Chammah (1966). It is a two-player symmetric game, where each player chooses either "Dove" or "Hawk," and the player's payoffs depend on her own action and that of the opponents as follows:

	Dove	Hawk
Dove	30,30	20,70
Hawk	70,20	0,0

Table 1. The Dove and Hawk Game.

A random $level_0$ chooses to play either Dove or Hawk with equal probability. A $level_1$ best responds to the $level_0$ player by choosing Hawk. A $level_2$ best responds to the $level_1$ player by

[7] There is a rich psychology literature supporting the finding of FCE or the "self-anchoring" argument. Mullen et al. (1985) reported 115 studies that show FCE. For more detailed empirical and theoretical discussion, refer to Marks and Miller (1987) and all the listed references therein.

choosing Dove, a level_3 best responds to the level_2 player by choosing Hawk, and so on. There are two pure NE strategies: (Hawk, Dove) and (Dove, Hawk) and a third mixed strategy where Dove and Hawk are played with are played with the probability of (1/3) and (2/3) respectively. Now, consider an NLK_1 player who faces a naive random level_0 player with the probability of λ and another NLK_1 player with the probability of $(1-\lambda)$. For $\frac{2}{3} \leq \lambda \leq 1$, only one pure strategy NLK equilibrium exists, where each player chooses Hawk. For $0 \leq \lambda < \frac{2}{3}$, there exist two pure strategy NLK equilibria: (Hawk, Dove) and (Dove, Hawk) and a mixed-strategy where Dove and Hawk are played with the probabilities of $\frac{2-3\lambda}{6(1-\lambda)}$ and $\frac{4-3\lambda}{6(1-\lambda)}$, respectively.

As a new solution concept, NLK shares a similar foundation to NE but is also applicable to games with players of different cognitive or reasoning abilities. For example, in the experiment of Alaoui and Penta (2016), math and science students who interact with students from humanities, may adopt a different subjective λ than when they play with fellow math and science. Such a conjecture, (e.g., larger λ) is reasonable and can be tested. We also adapt our basic definition of NLK to Bayesian games and dynamic games, as extensions of Bayesian Nash Equilibrium (BNE) and Subgame Perfect Nash Equilibrium (SPNE).

We are able to compare the performance of NLK to that of NE and some versions of Level-K by applying it to data from three experimental papers published in top economic journals and to data from a field study. These studies allow us to test the NLK on a static game of complete information and another with incomplete information, a dynamic game of perfect information, and on field data. For those experiments that we analyzed, NLK provides several insightful implications. First, in the static Guessing Game by Arad and Rubinstein (2012), a simple version of NLK with one parameter, $\lambda \in (0,1)$, that is chosen optimally, fits data better than both NE and Level-K models with an optimal distribution among three types of players, i.e., two parameters. Allowing for an error structure that is sensitive to payoffs, but using only one parameter, NLK still outperforms Level-K models. However, allowing Level-K to choose freely more parameters, fits better than the simple NLK, suggesting that in some cases, NLK can also serve as an analytical tool. Second, in application to the data from an experiment of the Centipede Game by Palacios-Huerta and Volij (2009), NLK's predictions, adapted to dynamic games, are different and more precise than those of SPNE and Level-K models, with only few exceptions when they coincide, or when Level-K adopts more parameters. It is also reassuring that the optimal $\lambda < 1$ is the largest when players are all students, the smallest when only chess players are involved, and in the middle when a chess player is matched with a student. Thus, the optimal λ for NLK seems to track and capture the shift in subjective beliefs that can be expected in the different mixes of subjects' populations. The better performance of NLK than Level-K in the Centipede Game is reconfirmed by using the data from Levitt, List, and Sadoff (2011). Moreover, NLK can capture belief updating in every round of a game that a dynamic Level-K cannot. Notably, although the results of data from the Centipede Game in the two aforementioned papers are drastically different, NLK predicts both quite well with different optimal λs, which implies the difference in behavioral data can be explained by the difference in beliefs of subjects between two datasets. In addition, we compare predictions of NLK to those of Level-K for the data from the Common-Value Auction experiment by Avery and Kagel (1997). For inexperienced bidders, NLK's performance coincides with that of Level-K; but for experienced bidders, NLK with $\lambda \in (0,1)$ provides the most accurate prediction. Moreover, since the estimated λ is larger for the data of experienced bidders than that of inexperienced bidders, NLK may also be used to track dynamic learning from experience, for example, learning in repeated games and convergence to a "full-equilibrium,"$\lambda = \rho = 0$. Finally, in a recent experimental work on a rank-order tournament with an outside option-a dynamic game with imperfect information, Brünner (2018) finds that a mixture of Level-K and NLK predicts both the population of types in the tournament, as well as the mean variance of efforts remarkably well. In fact, that paper show that NLK predicts the experimental data better than a level-K model without updating of beliefs, which highlights the importance of the belief updating that PBNLK added onto Level-K and the validity of NLK for outside sample predictions.[9]

Level-K and its related extension, Cognitive Hierarchy models by Camerer, Ho and Chong (2004) are applied to many laboratory experiments and field data. The survey by Crawford, Costa-Gomes, and Iriberri (2013) documents many successes of Level-K and its extensions over other solution concepts, including NE. However, as we saw in the Chicken Game above and several examples in the following paper, NLK can be more useful than Level-K in certain games.

Theoretically, Level-K has been extended in two ways. Strzalecki (2014) allows beliefs to vary arbitrarily for players at a certain level. Specifically, a level$_k$ player can believe the opponent

[9] Brünner (2018) shows that Nash Equilibrium performs even worse than the Level-K without belief updating.

to be level $j < k$, by any arbitrary subjective distribution. However, here as well, beliefs are restricted to lower levels. Building on Strzalecki (2014), Jimenez-Gomez (2018) innovatively allows a level$_k$ player to believe the opponent to be also a level$_k$ player, but only when their beliefs coincide, and adopts the solution concept of *interim correlated rationality* to games of incomplete information that endogenize level-0 behavior. However, in the application to the *e-mail game*, the case where the player allows the opponent to be the same level is not considered. Alaoui and Penta (2015) use another approach and show how cognitive bounds, beliefs about opponents, and beliefs about opponents' beliefs, vary according to incentives by a cost-benefit analysis. In their model, if agents believe that their opponents behave at lower levels than their own cognitive bound, they would behave at one level higher than these opponents; but if they believe that the strategies of their opponents are reaching or exceeding their own cognitive bound, they would act at their own cognitive bound. So, although the above researchers considered a situation where the opponents have the same or even a higher cognitive level than the agents themselves, they treated it as if the opponents were nevertheless one level below the agents. Thus, as far as we are aware, no extension of the Level-K model either allows the player to believe she faces the same level as herself, or applied such belief structure, in analysis of games.

NLK is not the first equilibrium solution concept to introduce an exogenous type; Kreps and Wilson; Milgrom and Roberts; and Kreps, Milgrom, Roberts, and Wilson (KMRW, all three papers were published in 1982), have already used an exogenous type. However, NLK and KMRW's models are different drastically in motivation and generality.

Motivation: KMRW's works are motivated by Selten's (1978) Chain-Store Paradox (CSP) and by vast experimental evidence of cooperation in finitely repeated Prisoner's Dilemma (PD) games. *Deterrence strategy* in CSP[10] and *cooperation* in PD games contradict the logic of backward induction that implies unraveling to the one-shot, stage game, solution. KMRW's objective is to resolve the paradoxes of using deterrence strategy in the CSP game and cooperation in the finitely repeated PD game. To do so, they transform these complete, into incomplete, information games by introducing a "tiny" probability of exogenous type and showing that it is sufficient to "choke off" the otherwise unavoidable logic of unraveling. The emphasis on tiny

[10] *Deterrence strategy*, where the monopoly fights an early entrant, although it is not the best response in the stage game, was offered by Selten (1978), as a sensible, though not an equilibrium, strategy to deter later entrant.

probability is a critical novelty, as otherwise deterrence strategy in the CSP game or cooperation in the PD game may be rationalized even in a one-shot game. In NLK, the probability λ of such exogenous type is typically quite large, similarly to, but may be smaller than, that in Level-K model. Thus, whereas the motivation of the KMRW's models is to "defend" the standard NE, NLK is a behavioral model of bounded rationality.

Generality: NLK introduces one nonstrategic exogenous type to be applied to all, or at least to a large class of different, games. In contrast, KMRW admit that their "defense" of the standard NE, requires a particular exogenous type for each case.[11] For instance, in the CSP case, Kreps and Wilson, use a "strong" monopoly, who is hard-wired to fight; in finitely repeated PD game, KMRW use two nonstrategic types for two cases respectively: the one who plays Tit-for-Tat in the one-sided incomplete information game, and the one who prefers the stage payoffs from joint cooperation over those of defection when the other player cooperates, for their two-sided case.[12]

Like other models that use "relaxed beliefs," NLK has its limitations. For example, NLK cannot explain deviations from theoretical predictions in games with a dominant strategy solution, such as overbidding one's value in Second Price Sealed Bid auctions with private values, first reported by Kagel, Harstad, and Levin (1987).

In Section 2, we present our basic solution concepts as used in different types of games (static or dynamic, with complete or incomplete information). In Section 3, 4 and 5, we provide the NLK solutions and compare them to those of NE and Level-K models for a static Guessing Game, a dynamic Centipede Game, and a Common Value Auction. We conclude in Section 6. Readers can refer to Appendix A.3 if interested in a more comprehensive literature review comparing related solution concepts.

[11] KMRW explicitly acknowledge that such particular, and different, exogenous type may be needed for different cases.

[12] In addition, NLK can require that λ matches the probability of the exogenous type in the population making the model an equilibrium model with rational expectations.

2. The solution concept

In this section, we formally define the NLK equilibrium in different types of games and prove its existence. We focus on the simple case of only two players with symmetric beliefs. We discuss several extensions in Section 5.

2.1 The Basic Case

Consider a normal form game with two players, $G = (S_i, u_i)_{i=1,2}$, where (S_i, u_i) are the strategy set and the utility function of player i, respectively. The strategy of a naïve player i is given exogenously by $\sigma_i^o \in \Delta(S_i), i = 1,2$. In our strategic environment, an NLK player believes that her opponent is either a naïve player with probability λ or another NLK player with probability $(1 - \lambda), \lambda \in [0, 1]$. In an NLK equilibrium, an NLK player chooses an optimal strategy by best responding to her belief. A formal definition of the NLK equilibrium is as follows:

Definition 1. *A mixed-strategy profile* $(\sigma_i^*)_{i=1,2}$, *is a* λ*-NLK equilibrium if for each* $i = 1,2$, *and each* $s_i' \in S_i$,

$$\lambda u_i(\sigma_i^*, \sigma_{-i}^0) + (1 - \lambda) u_i(\sigma_i^*, \sigma_{-i}^*) \geq \lambda u_i(s_i', \sigma_{-i}^0) + (1 - \lambda) u_i(s_i', \sigma_{-i}^*).$$

2.2. Bayesian Games

Consider a *Bayesian Game* of incomplete information $B = (S_i, u_i, \Theta_i, p)_{i=1,2}$, where Θ_i denotes the set of player i's types and where p is the joint *density function* of the probability distribution over $\Theta_1 \times \Theta_2$. Similar to the relationship between NE and BNE, a BNLK equilibrium is the NLK equilibrium of the "extended game" in which each player i's space of pure strategies is ${S_i}^{\Theta_i}$, which denotes the set of mappings from Θ_i to S_i. Again, let $\sigma_i^0 \in \Delta(S_i), i = 1,2$, denote the strategy of a naïve player i which is independent of his type. Then a formal definition of BNLK with respect to subjective symmetric belief λ is as follows:

Definition 2. *A profile of strategies* $\{s_i^*(\cdot)\}_{i=1,2}$, *is a* λ*-BNLK equilibrium, if for each* $i = 1,2$, *and each* $\theta_i \in \Theta_i$,

$$s_i^*(\theta_i) \in \arg max_{s_i \in S_i} \int p(\theta_{-i}|\theta_i)[\lambda u_i(s_i, \sigma_i^0; \theta_i, \theta_{-i}) + (1 - \lambda) u_i(s_i, s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i})] d\theta_{-i}.$$

2.3. Dynamic Games

Consider a dynamic game with perfect information and perfect recall played by two players[13] $P = (u_i, \Upsilon)_{i=1,2}$, where Υ denotes a game tree. A node in the game tree Υ is denoted by h^t, and the set of nodes is denoted by H. The set of nodes at which player i must move is denoted by H_i. An NLK player holds a prior belief that the opponent is either a naïve player with probability λ or another NLK player with probability $(1-\lambda), \lambda \in [0,1]$. At every decision node with history h^t, as more information is revealed, beliefs are updated. We denote the updated belief that the opponent is a naïve player as $p_i(h^t)$. In equilibrium, an NLK player chooses an optimal strategy according to her belief at every decision node. In other words, here the choice is sequentially rational as defined below:

Definition 3. (sequential rationality). *A strategy profile* $\{\sigma_i^*\}_{i=1,2}$ *is sequentially rational with respect to the profile of beliefs* $\{p_i(h_i^t)\}_{h_i^t \in H_i}, i = 1,2$ *if for* $i = 1,2$, *all strategies* σ_i', *and all nodes* $h_i^t \in H_i$*:*

(1) $$p_i(h_i^t)u_i(\sigma_i^*, \sigma_{-i}^0|h_i^t) + \left(1 - p_i(h_i^t)\right)u_i(\sigma_i^*, \sigma_{-i}^*|h_i^t) \geq p_i(h_i^t)u_i(\sigma_i', \sigma_{-i}^0|h_i^t) + (1 - p_i(h_i^t))u_i(\sigma_i', \sigma_{-i}^*|h_i^t).$$

We also require that the beliefs of an NLK player are consistent. That is, to start with a subjective prior distribution and then get updated by *Bayes' Rule* at each succeeding decision node. To present formally the consistency restriction, let $p(h^t|\sigma_i, \sigma_{-i})$ denote the probability that decision node h_t is reached according to the strategy profile, (σ_i, σ_{-i}).

Definition 4. (consistency). *A profile of beliefs* $\{p_i^*(h_i^t)\}_{h_i^t \in H_i}, i = 1,2$ *is consistent with the subjective prior* λ *and the strategy profile* $\{\sigma_i\}_{i=1,2}$ *if and only if for i=1,2, and all nodes* $h_i^t \in H_i$*:*

(2) $$p_i^*(h_i^t) = \frac{\lambda p(h_i^t|\sigma_i, \sigma_{-i}^0)}{\lambda p(h_i^t|\sigma_i, \sigma_{-i}^0) + (1-\lambda)p(h_i^t|\sigma_i, \sigma_{-i})},$$

Where, $p(h_i^t|\sigma_i, \sigma_{-i}^0) > 0$ or $p(h_i^t|\sigma_i, \sigma_{-i}) > 0$.[14]

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[13] That is, at any decision node, all previous moves are assumed to be known to every player.

[14] It should be noted that Definition 1.4 places no restrictions on player i's expectations about those decision nodes that are not possibly reached according to σ, regardless of facing a naïve player or another NLK player. A stronger notion of consistency could be defined in the spirit of a trembling hand or a sequential equilibrium (Kreps and Wilson, 1982a). Such stronger restriction and its impact on prediction are discussed in Section 5.

Although the game itself has perfect information, the belief structure in our strategic environment makes our solution concept more like an analogy of a *Perfect Bayesian Equilibrium* (PBE), So we denote it as PBNLK, formally treated below:

Definition 5. An assessment $\left(\sigma_i^*, \{p_i^*(h_i^t)\}_{h_i^t \in H_i}\right)_{i=1,2}$ is a *λ-PBNLK* equilibrium if

1. The strategy profile $\{\sigma_i^*\}_{i=1,2}$ is sequentially rational with respect to the profile of beliefs $\{p_i^*(h_i^t)\}_{h_i^t \in H_i}, i = 1,2$ and
2. The profile of beliefs $\{p_i^*(h_i^t)\}_{h_i^t \in H_i}, i = 1,2$ is consistent with the subjective prior λ and the strategy profile $\{\sigma_i^*\}_{i=1,2}$.

2.4. Existence

Proposition 1. for any $\lambda \in [0, 1]$:

a) In every finite strategic-form game, there exists an NLK equilibrium.
b) In every finite Bayesian game, there exists a BNLK equilibrium.
c) In every finite extensive form game, there exists a PBNLK equilibrium.

Proof: The existence of an NLK equilibrium is guaranteed by a standard fixed-point theorem (Kakutani 1941), similar to the proof of the existence of a NE (Glicksberg 1952). For (b), a similar argument follows from Harsanyi (1973). For (c), consider an alternative dynamic game of incomplete information, $\mathrm{P} = (\mathrm{u_i}, \Phi_\mathrm{i}, \Upsilon)_{\mathrm{i}=1,2}$, where Φ_i denotes the possible types for agent i, which can be either a naïve player or an NLK player. Let Σ^0be the strategy set of the naïve player and restrict it to be $\{\sigma^0\}$. Then according to Kreps and Wilson (1982a), for every finite extensive form game, there exists at least one sequential equilibrium $(\sigma_i^*, p_i^*)_{i=1,2}$ should satisfy equation 1 and 2 for sequential rationality and consistency. In other words, PBNLK exists.

Remark 1.1. In the special case of $\lambda = 0$, NLK/BNLK/PBNLK coincides with NE/BNE /SPNE. In the special case where the strategy of the naïve player is exogenously given as that of a level$_{k-1}$ player, where $\mathrm{k} \in \aleph^+$ and $\lambda = 1$, then the strategy for an NLK player coincides with that of a level$_k$ player in all three types of games considered above.

3. The Arad-Rubinstein Money Request Game.

In the basic version of the Money Request Game by Arad and Rubinstein (2012), there are two risk-neutral players, and each can request and receive an integer amount of money from $11 to $20, plus an extra $20 if she asks for exactly one integer less than the other player.

NLK (%) (λ)	15	16	17	18	19	20
$[0 \leq \lambda \leq \frac{1}{2})$	$\frac{5(5-10\lambda)}{1-\lambda}$	$\frac{5(5-2\lambda)}{1-\lambda}$	$\frac{5(4-2\lambda)}{1-\lambda}$	$\frac{5(3-2\lambda)}{1-\lambda}$	$\frac{5(2-2\lambda)}{1-\lambda}$	$\frac{5(1-2\lambda)}{1-\lambda}$
$[\frac{1}{2} \leq \lambda \leq \frac{14}{20})$	0	$\frac{5(14-20\lambda)}{1-\lambda}$	$\frac{15}{1-\lambda}$	$\frac{10}{1-\lambda}$	$\frac{5}{1-\lambda}$	0
$[\frac{14}{20} \leq \lambda \leq \frac{17}{20})$	0	0	$\frac{5(17-20\lambda)}{1-\lambda}$	$\frac{10}{1-\lambda}$	$\frac{5}{1-\lambda}$	0
$[\frac{17}{20} \leq \lambda \leq \frac{19}{20})$	0	0	0	$\frac{5(19-20\lambda)}{1-\lambda}$	$\frac{5}{1-\lambda}$	0
$[\frac{19}{20} \leq \lambda \leq 1]$	0	0	0	0	100	0

Table 2. NLK equilibrium strategy for different subjective beliefs.

Consider the Level-K model with a level_0 payer who randomizes uniformly within the strategy set: $\{\$11, \$12, \ldots, \$20\}$. A level_1 player that requests \$20 earns \$20. Alternatively, if she asks for \$19, she would earn \$19 for sure and \$20 bonus with a probability of 1/10, for a total expected payoff of \$21.[15] Thus, level_1 picks \$19, level_2 picks \$18,…, and level_9 picks \$11. But then level_{10} picks \$20, level_{11} picks \$19, and so on. It is difficult to infer from players' actions their sophistication level: A player who requests \$19 can be a level_1 player or a highly sophisticated level_{11} player.

Table 2 shows the unique mixed strategy λ-NLK equilibrium for each $\lambda \in [0,19/20)$ and the unique pure strategy for $\lambda \in [19/20,1]$.[16]

Table 3 compares the performance of the Level-K model, $k = 1,2,3,$ NE, and NLK[17] by using the Mean Squared Error (MSE). NLK with the best λ $(= 0.6585)$ fits the data better than NE and any type of the Level-K model; moreover, it also outperforms Level-K with the optimal distribution of level_1, level_2 and level_3, i.e., two parameters, as it reduces MSE by 23.45%. (From MSE=35.93 to 28.39.)

[15] To ask for any amount of money less than \$19 lead to s strictly lower payoff.
[16] See Appendix A.1 for detail of the argument.
[17] Our naïve player is defined in the same way as the random level_0 player.

Action	11	12	13	14	15	16	17	18	19	20	MSE
Level_1 (%)	0	0	0	0	0	0	0	0	100	0	980.2
Level_2 (%)	0	0	0	0	0	0	0	100	0	0	620.2
Level_3 (%)	0	0	0	0	0	0	100	0	0	0	580.2
$level_k, k = 1,2,3,$ $optimal\ distribution$	0	0	0	0	0	0	40.7	38.7	20.6	0	35.93
NE (%)	0	0	0	0	25	25	20	15	10	5	137.2
NLK (%)[18] $(\lambda = 0.6585)$	0	0	0	0	0	12.1	43.9	29.4	14.6	0	28.39
Data (%)	4	0	3	6	1	6	32	30	12	6	

Table 3. 11-20 Game: Comparison of different solution concepts by MSE.

Finally, we further test the robustness of our results by an alternative statistical method. Our econometric specification follows the mixture-of-types models of Stahl and Wilson (1994, 1995).[19]

Both level_k and our NLK types are assumed to make logistic errors as described below. The decision rule suggests that the choice probabilities of type t players are positively, but imperfectly, related to expected payoffs according to the specific beliefs of type t. Formally, denote the expected payoff player i of type t, given strategy s by $\pi_i^t(s)$. Then, the probability of observing s by such players is specified as follows:

$$p_i^t(s) = \frac{\exp\,(\eta\pi_i^t(s))}{\sum_{s'\in S_i}\exp\,(\eta\pi_i^t(s'))},$$

where S_i is the strategy set for player i and η is the parameter of precision. Specifically, η determines the sensitivity of choice probabilities to payoff differences.[20] Exceptionally, random level_0 directly specifies a uniform distribution of decisions, thus has no precision parameter. Alternatively, it is equivalent by specifying the precision parameter to be 0 for a random level_0 player. The *Likelihood* of observing sample $\{s_i\}_{i=1}^N$, given type t is $L^t(\{s_i\}|\eta) = \prod_{i=1}^N p_i^t(s_i)$.

[18] We choose the value that minimizes the Mean Squared Errors (MSE), that is, the nonlinear least squares estimate of λ.

[19] The same econometric specification was also adopted by Costa-Gomes, Crawford and Broseta (2001), Camerer, Ho and Chong (2004), Costa-Gomes and Crawford (2006) and Crawford and Iriberri (2007). The error model is developed from *Quantal Response Equilibrium* (See, e.g., Goeree, Holt and Palfrey 2008), and discussed in Goeree and Holt (2001).

[20] As η goes to ∞, the probability of the optimal decision converges to 1. In other words, the choice is error-free and fully characterized by the model under consideration. At the other extreme, as η goes to 0, the choice probability converges to a uniformly random choice as that of the random level_0 players.

Let α_t denote the proportion of type t in the population, with $\sum_t \alpha_t = 1$. The *Likelihood* of observing the sample unconditional on type is $\prod_{i=1}^{N} \sum_t \alpha_t p_i^t(s_i)$.

Action	Log-Likelihood (LL)	The precision parameter (η)	BIC[21]	AIC[22]
Level_1	-233.970	0.296 (0.039)	472.622	469.940
Level_2	-226.245	0.066 (0.009)	457.172	454.490
Level_3	-221.220	0.075 (0.010)	447.122	444.440
$level_k, k = 1,2$ $optimal\ distribution$	-218.100	0.252 (0.052)	445.564	440.200
$level_k, k = 1,2,3,$ $optimal\ distribution$	-197.770	0.207 (0.051)	409.586	401.540
NE	-230.040	0.231 (0.046)	464.762	462.080
NLK ($\lambda = 0.85$)	-210.050	0.359 (0.025)	429.464	424.100

Table 4. Comparison of different solution concepts by Maximum Log-Likelihood

The results are reported in Table 4. With an error structure, the best single type Level-K model with $k^* = 3$ has a smaller Log-likelihood and a precision parameter, $LL = -221.275$, $\eta = 0.075$ than those of NLK with the best $\lambda^* = 0.85$: $LL = -210.05, \eta = 0.359$. NLK also outperforms Level-K model with the optimal distribution of level_1 and level_2:[23] $LL = -218.093$, $\eta = 0.252$. But, letting Level-K use two parameters, and optimal distribution, of level_1 level_2 and level_3, raises its LL to, -197.77, which is larger than that of NLK with only one parameter, -210.05. Yet, NLK still has a higher precision, $\eta = 0.231$, than that of Level-k, η =0.207.[24] The results are robust when consider BIC and AIC instead of LL.

[21] $BIC = kln(n) - 2LL$, k is the number of free parameters to be chosen and n is the number of observations.
[22] $AIC = 2k - 2LL$, k is the number of free parameters to be chosen
[23] It is estimated to be 85% level_1 and 15% level_2 types.
[24] It is estimated to be 46% level_1, 24.45% level_2 and 28.98% level_3 types.

4. The Centipede Game

Introduced by Rosenthal (1981), the Centipede Game is an example where deviations from *Backward Induction* (or SPNE) seem reasonable.[25] Following the bulk of the literature, we study a version of the Centipede Game, where the total payout doubles when the game continues to the next stage, which subsumes the game in the experiment of both Palacios-Huerta and Volij (2009) and Levitt, List, and Sadoff (2011), as a special case (with six decision nodes).

There are two players, A and B, with an initial pot worth $5. At Node 1, Player A moves and chooses either to stop the game (T) by taking 80% of the pot and leaving 20% of it to Player B, or pass the game (P) to Player B and doubling the pot. If Player A chooses P, then at Node 2, Player B faces a similar decision but with a pot now worth $10. Unless one of the players chooses T earlier, the game ends after S=2N stages, with Player B either choosing T, taking 80% of the pot and leaving the other 20% to Player A, or choosing P and doubling the pot, with the result that 20% of the pot goes to Player B and 80% of it goes to Player A. The payoffs for Players A and B are ($\$2^{2k}, \$2^{sk-2}$) if the game ends at an odd decision node, $2k-1$, and ($\$2^{2k-1}, \$2^{sk+1}$) if the game ends at an even decision node, $2k$, $k = 1, 2, \dots, N-1$. By backward induction, the unique SPNE strategy profile for Player A is to play T immediately, at node 1, and off equilibrium, the active player always chooses T at each node.

Following the dynamic Level-K model by Ho and Su (2013), it is equally likely that a level$_0$ player will choose T or P at each decision node, and strategies of $K > 0$ are generated from iterative best responses to a player of one level below. Level$_1$ Player B would choose T at the last node.[26] Denote the whole pie at each decision node by x. A level$_1$ Player A, playing T at node $(2N-1)$ yields $\frac{4x}{5}$, where playing P yields $\frac{9x}{5}$, so a level$_1$ Player A would choose P at the decision node $(2N-1)$. Similarly, a level$_2$ Player A would choose T at the penultimate node (2N-1).

Table 5 summarizes the solution for the Level-K model for a game of length $S = 2N$. For a certain level of players (indicated in the second column), there exists a corresponding threshold stage (indicated in the first column). A level$_k$ player chooses P before the threshold stage s^*, but T at stage s^* and afterward. For example, in a six-stage game ($N = 3$), the threshold stage for a

[25] For additional literature, see McKelvey and Palfrey (1992), Fey, McKelvey, and Palfrey (1996), Nagel and Tang (1998), Borstein, Kugler, and Ziegelmeyer (2004), and Rapoport, Stein, Parco, and Nicholas (2003). These papers show that even in high-stakes situations, involving altruism or group decisions, *Backward Induction* is still inadequate to explain players' behavior.

[26] To end the game at Node $2N$, Player B gets payoff $\$2^{2N+1}$, while he only ends up with $\$2^{2N}$ if he chooses P instead

level$_3$ $(k=3, h=1)$ Player A is $2(3-1)+1=5$. So, a level$_3$ Player A chooses P before Node 5 and T at Node 5.

Role	Threshold stage s	The level of players
Player A	$2(N-h)+1$	$k=2h^*$ or $2h+1$ $(1 \le h \le N-1)$
Player B	$2(N-h)+2$	$k=2h^*-1$ or $2h$ $(1 \le h \le N)$

h^* is an auxiliary parameter for indicating the same threshold stage of two adjacent levels.

Table 5. Threshold stage for different levels of players.

In addition, a Player A, at level $k=2N$ or higher, and a Player B at $k=(2N-1)$ or higher, ought to choose T at each decision node. The Level-K solution requires relatively high levels[27] to rationalize terminating the game at earlier stages, especially for longer games, since the strategies of different level players are independent of the length of the game. For example, no matter how long the game is, a level$_1$ Player A ought to keep passing to the last decision node, and no matter what the observed history is, a level$_k$ player never updates his belief. [28]

Consider a simple version of PBNLK with symmetric beliefs, $0<\lambda<1$. At the last stage, T is the best response for Player B regardless of his belief about his opponents' type. Assume now that Player B first chooses T at Stage $2n$ and Player A plans to choose T at Stage $(2n+1)$. Then, at stage $(2n-1)$, Player A's posterior belief of the opponent being level$_0$ is

$$(3) \qquad p_A^\lambda(2n-1)=\frac{\lambda\left(\frac{1}{2}\right)^{n-1}}{\lambda\left(\frac{1}{2}\right)^{n-1}+(1-\lambda)}=\left[\frac{\left(\frac{1}{2}\right)^{n-1}}{\left(\frac{1}{2}\right)^{n-1}+\frac{(1-\lambda)}{\lambda}}\right]\in(0,\lambda).$$

If Player B first plays T at Stage $2n$, then at Stage $(2n-1)$, Player A gets $\frac{4x}{5}$ by playing T, while by playing P now and T at $(2n+1)$ yields the expected payoff:

$$\left[\frac{p_A^\lambda(2n-1)}{2}+1-p_A^\lambda(2n-1)\right]\frac{2x}{5}+p_A^\lambda(2n-1)\frac{1}{2}\times\frac{4}{5}\times 4x=\frac{2x}{5}+p_A^\lambda(2n-1)\frac{7x}{5}$$

Thus, Player A plays P whenever $\frac{2}{7}<p_A^\lambda(2n-1)\le 1$ and plays T otherwise. Moreover, since $p_A^\lambda(2N-1)$ decreases in N for a given λ, then, in a longer game, and NLK Player A (with a

[27] Table 5 also entails that to increase the level by just 1 would not necessarily predict earlier termination. Two adjacent levels of players might behave the same way.

[28] One may argue that the more general Cognitive Hierarchy (CH) solution concept would produce qualitatively different predictions. However, since beliefs put more weight on lower levels according to a Poisson distribution in CH and lower levels continue passing to later stages, an even higher level of players than in the Level-K model would be required to rationalize early termination.

certain λ) is more likely to play T at stage $(2N-1)$. This result is a key difference between NLK and the Level-K model where a level$_1$ Player A always passes at stage $(2N-1)$ no matter how long the game is. Since $p_A^\lambda(2n-1)(\leq\lambda)$ is strictly decreasing in n, and $p_A^\lambda(2n-1)_{n\to\infty}=0$, then for $\lambda\leq\frac{2}{7}$, Player A would always play T, given that Player B plays T at the next sage. For $\lambda>\frac{2}{7}$, by continuity, there is a critical value n_A, such that $p_A^\lambda(2n-1)>\frac{2}{7}$ for $n<n_A$, and $p_A^\lambda(2n-1)\leq\frac{2}{7}$ for $n\geq n_A$.

Similarly, assume that Player A first chooses T at Stage $(2n+1)$, $(n\leq N-1)$ and Player B plans to choose T at stage $(2n+2)$. Then at Stage $2n$, Player B's posterior belief that the opponent is level$_0$ is

$$p_B^\lambda(2n)=\frac{\lambda\left(\frac{1}{2}\right)^{n-1}}{\lambda\left(\frac{1}{2}\right)^{n-1}+(1-\lambda)}=p_A^\lambda(2n+1).$$

This implies that the threshold stage for Player B, s_B^* is one stage earlier than that of Player A, s_A^*, i.e., $s_B^*=(2n_B)=(2n_A-1)-1=s_A^*-1$.

We use these arguments to construct our PBNLK equilibrium. For $\lambda=0$, the game ends at the first stage (the same result as in SPNE).[29] For $\lambda>0$, there are two possibilities. In a short game with a relatively larger λ satisfying $p_A^\lambda(2N-1)>\frac{2}{7}$, Player A plays P to the end, and Player B first plays T at the last stage (the same result as when both players are level$_1$). In a longer game with $p_A^\lambda(2N-1)\leq\frac{2}{7}$, the game would end earlier. For similar arguments as in Kreps, Milgrom, Roberts, and Wilson (1982) paper,[30] PBNLK must be in mixed strategies for this range of λ. The reason is that in a presumed pure strategy PBNLK, and NLK player (who ought to play T earlier than the other player) would rather deviate in the first node she ought to play T and lay P instead. Doing so would mislead the other player to believe that he is facing a level$_0$ player (as only a level$_0$ player would have played P in the last node) and thus the other NLK player would play P.[31]

[29] The off-equilibrium path will not be reached by an NLK player (A or B) whether her opponent is another NLK player or a naive player. So, it is not restricted by Definition 1.4 of consistency. We assume that an NLK player believes the other NLK player would always play T off the equilibrium path.

[30] Inserting a "crazy" type even with a slight probability can rationalize long cooperation in the finitely repeated prisoners' dilemma games.

[31] For example, consider the case when the threshold stage of Player B is 4 and (it follows) that of Player A is 5, Now at Node 5, which is reachable for Player A when facing a level$_0$ player, since Player B first choose T at 4, not 6, the belief $P_A^\lambda(5)$ represented by Equation 1.3 no longer satisfies our consistency requirement. Upon reaching Node 5, by

We apply our model to experimental by Palacios-Huerta and Volij (2009) and Levitt, List, and Sadoff (2011) on the *Centipede Game*, where *N=3* as in Figure 1.

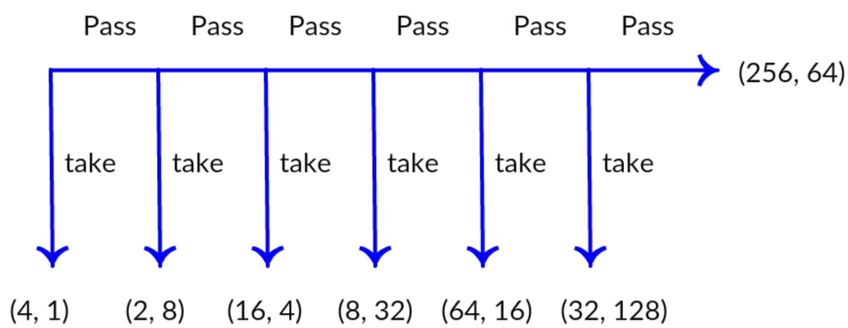

Figure 1. The Centipede Game[32]

The prediction of our PBNLK with all $\lambda \in \{0.05n\}_{n=0,1,2,\dots,20}$ and the Level-K model with all $k \in \aleph^{+}$ as well as data from the above-mentioned two papers are summarized in Table 6. When $\lambda = 0$, PBNLK coincides with SPNE and level_k, $k \geq 6$, and when $\lambda \in [0.615, 1]$, PBNLK coincides with level_1. For all other $\lambda \in (0, 0.615)$, PBNLK generates different predictions. We first compare predictions to data from Palacios-Huerta and Volij's laboratory experiment with four treatments. Unlike other experiments of the *Centipede Game*, in their work, the composition of two opponents varies[33] across treatments, and it is common knowledge among all players. This allows us to explore how belief represented by λ and the results change as the nature of the subject pool changes. Next, we compare predictions to data from Levitt, List, and Sadoff's field experiments of chess players to further evaluate the predictions of NLK, since the data are quite different from the former experiment data.

Bayes' rule, an NLK Player A confirms that her opponent is a level_0 player for sure, so she would pass instead. Thus, at decision Node 4, an NLK Player B has an incentive to pass with a positive probability to mimic the level_0 player which motivates an NLK Player A to pass with a positive probability at decision Node 5, as well.

[32] This is the same example from Palacios-Huerta and Volij (2009). Source: Drawn using Microsoft Visio.

[33] See Table 6 for detail. The two opponents are chess players or students.

Data or Prediction	*Node 1*	*Node 2*	*Node 3*	*Node 4*	*Node 5*	*Node 6*
NLK $(\lambda = 0)$ or $level_k$ $(k \geq 6)$	1*	1	1	1	1	1
NLK $(\lambda = 0.05)$	0	0.704	0.867	0.899	0.892	1
NLK $(\lambda = 0.1)$	0	0.375	0.877	0.889	0.938	1
NLK $(\lambda = 0.15)$	0	0.007	0.889	0.889	0.999	1
NLK $(\lambda = 0.2)$	0	0	0	0.844	0.879	1
NLK $(\lambda = 0.25)$	0	0	0	0.792	0.887	1
NLK $(\lambda = 0.3)$	0	0	0	0.732	0.895	1
NLK $(\lambda = 0.35)$	0	0	0	0.663	0.905	1
NLK $(\lambda = 0.4)$	0	0	0	0.583	0.916	1
NLK $(\lambda = 0.45)$	0	0	0	0.489	0.930	1
NLK $(\lambda = 0.5)$	0	0	0	0.375	0.946	1
NLK $(\lambda = 0.55)$	0	0	0	0.236	0.966	1
NLK $(\lambda = 0.6)$	0	0	0	0.0625	0.991	1
NLK or $level_1$ $(0.615 < \lambda \leq 1)$	0	0	0	0	0	1

Continued

Table 6. Centipede Game-Prediction and Data

Table 6 Continued

Data or Prediction	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
$level_2$	0	0	0	0	1	1
$level_3$	0	0	0	1	1	1
$level_4$	0	0	1	1	1	1
$level_5$	0	1	1	1	1	1
Data** (S vs S)	0.030 *** (200)	0.17 (194)	0.42 (161)	0.65 (93)	0.82 (33)	0.83 (6)
Data (S vs C)	0.30 (200)	0.52 (140)	0.61 (67)	0.69 (26)	1.00 (8)	-
Data (C vs S)	0.375 (200)	0.44 (125)	0.56 (70)	0.61 (31)	1.00 (12)	-
Data (C vs C)	0.725 (200)	0.64 (55)	0.90 (20)	1.00 (2)	-	-
Data****(Field)	0.039 (102)	0.102 (98)	0.193 (88)	0.352 (71)	0.587 (46)	0.632 (19)

Note: * presents predicted probabilities of playing T at each node by the model. Columns correspond to the probability that a player is predicted to play T upon reaching that node. Odd nodes refer to Player A's choices, while even nodes refer to Player B's choices.
** The data is from Palacios-Huerta and Volij (2009). S represents students and C represents chess players. S vs C represents the situation when Player A is a student and Player B is a chess player. The other way around, C vs S is when Player A is a chess player and Player B is a student.
*** shows the distribution of implied stop probabilities for players in the Centipede Game. The number of opportunities observed is displayed in the parentheses below.
**** The data is from the field Centipede Game of chess players by Levitt, List, and Sadoff (2011).

Referring to Ho and Su (2013), we define a measure, $D(H, M, G_S)$, to quantify the deviation of data H from the model's prediction, M in *Centipede Game* G_S with S decision nodes as follows:

$$D(H, M, G_S) = \sum_{s=1}^{S} w_s^H d_s(p_s^H, p_s^M)\ ,\ w_s^H = \frac{n_s^H}{\sum_{k=1}^{S} n_k^H}\ ,\ d_s(p_s^H, p_s^M) = |p_s^H - p_s^M|,$$

where n_s^H is the number of observations at each stage, given by data H, $d_s(p_s^H, p_s^M)$ is the distance of stopping probabilities at stage s between data H and the prediction of model M measured by their absolute difference $|p_s^H - p_s^M|$.

Models	Data (S vs S)	Data (S vs C)	Data (C vs S)	Data (C vs C)	Data (Field)
NLK $(\lambda = 0)$ or $level_k$ (k ≥ 6)	0.7102	0.5474	0.5431	$0.2773^{\#*}$	0.7760
NLK $(\lambda = 0.05)$	0.3016	0.2478	0.3191	0.5393	0.4296
NLK $(\lambda = 0.1)$	0.2132	$0.2361^{\#}$	$0.2619^{\#}$	0.5785	0.3589
NLK $(\lambda = 0.15)$	0.2071	0.3536	0.3672	0.6500	0.3269
NLK $(\lambda = 0.2)$	0.1857	0.4051	0.4062	0.7166	0.2036
NLK $(\lambda = 0.25)$	0.1791	0.4019	0.4023	0.7170	0.1946
NLK $(\lambda = 0.3)$	0.1714	0.3982	0.3978	0.7175	0.1866
NLK $(\lambda = 0.35)$	$0.1625^{\#}$	0.3971	0.3927	0.7181	0.1761
NLK $(\lambda = 0.4)$	0.1703	0.4016	0.3905	0.7185	0.1638
NLK $(\lambda = 0.45)$	0.1837	0.4069	0.3968	0.7192	0.1497
NLK $(\lambda = 0.5)$	0.1999	0.4134	0.4044	0.7200	$0.1323^{\#}$
NLK $(\lambda = 0.55)$	0.2197	0.4212	0.4137	0.7210	0.150
NLK $(\lambda = 0.6)$	0.2444	0.4310	0.4253	0.7222	0.1818
NLK $or\ level_1$ $(0.615 < \lambda \leq 1)$	0.2840	0.4526	0.4569	0.7227	0.2121

Continued

Table 7. Centipede Game-Prediction for different models

Table 7 Continued

$level_2$	0.2533	0.4345	0.4295	0.7227	0.1933*
$level_3$	0.2127*	0.4121	0.4139	0.7155	0.2428
$level_4$	0.2502	0.3787*	0.3947*	0.6578	0.3703
$level_5$	0.4366	0.3660	0.4290	0.6022	0.5540
$level_k, k = 1,2$ $optimal\ distribution$	0.2446	0.4345	0.4295	0.7227	0.1484
$level_k, k = 1,2,3,$ $optimal\ distribution$	0.1567	0.3946	0.3872	0.7155	0.0895

Note: * and # indicate the best prediction of a single type Level-K and NLK, respectively.

Table 7 presents the result of $D(H, M, G_S)$ calculated using PBNLK and Level-K model with 5 different data sets above. In the lab experiments when opponents are students (Column 2), PBNLK with $\lambda = 0.35$ gives the most precise prediction ($D = 0.1625$), which is better than the best prediction of the single type Level-K model ($k = 3, D = 0.2127$); in the treatment when chess players and students play with each other (Column 3 and 4), PBNLK with $\lambda = 0.1$ fits the data the best ($D^{S\ vs\ C} = 0.2361, D^{C\ vs\ S} = 0.2619$), which is more accurate than the Level-K model with an optimal $k = 4$ ($D^{S\ vs\ C} = 0.3787, D^{C\ vs\ S} = 0.3947$); when the opponents are chess players, the best fit goes to the case when PBNLK ($\lambda = 0$), SPNE and the $level_k$ type, ($k \geq 6$) coincide ($D = 0.2773$). For the field data, PBNLK with $\lambda = 0.5$ gives the most precise prediction ($D = 0.1323$), which is more accurate than the best prediction of the Level-K model with an optimal $k = 2$ ($D = 0.1933$). Moreover, in all 5 datasets, the optimal PBNLK performs better than the Level-K with an optimal distribution of $level_1$ and $level_2$ types. When we allow the Level-K model to have one more parameter, the optimal PBNLK still performs better the Level-K with an optimal distribution of $level_1$, $level_2$ and $level_3$ in three datasets (Column 2, 3, 4) except for the lab experiments when opponents are students and the field data.

Our solution concept provides an alternative explanation for cases where neither the original Level-K model nor backward induction applies. Note that we constrained NLK by using only symmetric beliefs. However, it is reasonable for each group to have a different subjective λ in cases where students interact with chess players, wherein we conjecture that NLK would perform even better by allowing for heterogeneous beliefs, but accounting for additional parameters.

5. Common Value Auction

Avery and Kagel (1997, AK afterward) conducted a laboratory experiment using a Common-Value, Second-Price Auction, *the Wallet Game*. In their design, there are two bidders, $i = 1,2$, each privately observes a signal X_i that is drawn i.i.d from a *uniform distribution* on $[1, 4]$. The common-value is the sum of the two private signals, that is, $v_i(x_1, x_2) = v(x_1, x_2) = x_1 + x_2$. Let $v(x, y) = x + y$, and $\mathrm{r(x)} = \mathrm{x} + \mathrm{E}[X_2] = x + 2.5$. $v(x, x) \equiv b(x) = 2x$ is the unique symmetric BNE,[34] and with just two bidders, $b(x) = 2x$, is an *ex-post* equilibrium, independent of signals' distribution, risk attitude and with no regret. AK defines *Naïve bidding* by $r(x) = x + 2.5$, representing a naïve bidder who assumes that whenever she wins, the other bidder's signal is at its expected value (2.5). It turns out that $r(x)$ is also the level_1 player's strategy in Crawford and Iriberri (2007, CI afterward), the best response to a level_0 player who bids uniformly randomly on $[1, 4]$. We denote by $b^{\lambda}(\cdot)$ the strategy in a λ-BNLK equilibrium and solve the symmetric linear strategy. (The details are provided in Appendix A.2.)

The data produced by AK is evaluated using the CE model by Eyster and Rabin (ER, 2005) and the Level-K by Crawford and Iriberri (CI, 2007). ER show that for any cursed level, $0 < \chi \leq 1$, their CE predicts better than BNE (i.e. CE with $\chi = 0$) and that for a given χ, CE fits better for experienced, rather than for inexperienced subjects, with respect to the Mean Squared Error (MSE). For data on only inexperienced bidders, CI use the Level-K with a logistic error structure and a subject-specific precision. They compare their model using the best mixture of 5 types, including random level_1 and level_2,[35] truthful level_1 and level_2,[36] and BNE players and show that it outperforms CE (with the best mixture of types, such that $\chi \in \{0.1, 0.2, \ldots, 0.9, 1.0\}$), using both likelihood and the Bayesian Information Criterion (BIC).[37]

[34] Refer to Milgrom and Weber (1982).
[35] Random level_1 and level_2 are generated iteratively by best responding to a random level_0, as considered in this paper.
[36] Truthful level_1 and level_2 are generated iteratively by best responding to a truthful level_0 who always bids her signal: $b(x) = x$.
[37] BIC penalize models with more parameters to adjust the likelihood.

Models	$b(x)$	MSE (inexperienced)	MSE (experienced)
NLK $(\lambda = 0)$ or NE	2x	2.897	1.171
NLK $(\lambda = 0.05)$	1.951x+0.122	2.823	1.124
NLK $(\lambda = 0.1)$	1.904x+0.239	2.756	1.082
NLK $(\lambda = 0.15)$	1.859x+0.352	2.693	1.042
NLK $(\lambda = 0.2)$	1.815x+0.462	2.634	1.010
NLK $(\lambda = 0.25)$	1.772x+0.570	2.579	0.978
NLK $(\lambda = 0.3)$	1.730x+0.676	2.531	0.953
NLK $(\lambda = 0.35)$	1.688x+0.781	2.484	0.927
NLK $(\lambda = 0.4)$	1.646x+0.886	2.440	0.906
NLK $(\lambda = 0.45)$	1.604x+0.990	2.396	0.889
NLK $(\lambda = 0.5)$	1.562x+1.096	2.356	0.872
NLK $(\lambda = 0.55)$	1.519x+1.203	2.320	0.859
NLK $(\lambda = 0.6)$	1.475x+1.313	2.286	0.848
NLK $(\lambda = 0.65)$	1.430x+1.426	2.250	0.840
NLK $(\lambda = 0.70)$	1.383x+1.543	2.220	0.835
NLK $(\lambda = 0.75)$	1.333x+1.667	2.190	0.834*
NLK $(\lambda = 0.80)$	1.281x+1.798	2.164	0.835

Continued

Table 8. Model comparison for the wallet game.

Table 8 Continued

NLK $(\lambda = 0.85)$	1.224x+1.940	2.137	0.843
NLK $(\lambda = 0.90)$	1.161x+2.098	2.117	0.857
NLK $(\lambda = 0.95)$	1.088x+2.280	2.097	0.882
NLK $(\lambda = 1)$ or $level_1$	x+2.5	$2.085^{\#*}$	$0.922^{\#*}$
$level_2$	$\begin{cases} 3.5\ if\ x < 2.5 \\ 6.5\ if\ x > 2.5 \end{cases}$	2.955	1.381
$Level_3$	$\begin{cases} < 3.5\ if\ x < 2.5 \\ > 6.5\ if\ x > 2.5 \end{cases}$	-	
Data (inexperienced)	$\underset{(0.079)}{0.997}$x+$\underset{(0.203)}{2.950}$	1.899	
Data(experienced)	$\underset{(0.053)}{1.313}$x+$\underset{(0.150)}{2.023}$	-	0.745

Table 8 compares the prediction of λ-BNLK (with all $\lambda \in \{0.05\text{n}\}_{n=0,1,2,\dots,20}$) and the Level-K model. As shown, the optimal bidding of a $level_2$ player already reduces to a boundary solution (the objective function becomes a linear function), where all bidders with a value lower than 2.5 bid 3.5 and the others (with a value higher than 2.5) bid 6.5. For a $level_3$ player, when her signal is smaller than 2.5, she bids any number below 3.5, (expecting to lose), while bidding any number above 6.5 when her signal is larger than 2.5 (expecting to win). The predictions are ambiguous for higher levels. In contrast, there always exists a symmetric linear strategy for our λ-BNLK players.

Table 8, Figures 2 and 3, show that for inexperienced bidders (using the first 18 periods), the most accurate prediction of BNLK is with $\lambda = 1$, and it coincides with $level_1$ (MSE=2.085)[38]. For experienced bidders (using periods 19-42), BNLK with $\lambda = 0.75$ fits the data the best

[38] We choose the value of λ that minimizes the Mean Squared Errors (MSE), that is, the nonlinear least squares estimate of λ.

($MSE = 0.834$) which is better than the most precise prediction of the Level-K ($k = 1, MSE = 0.922$).

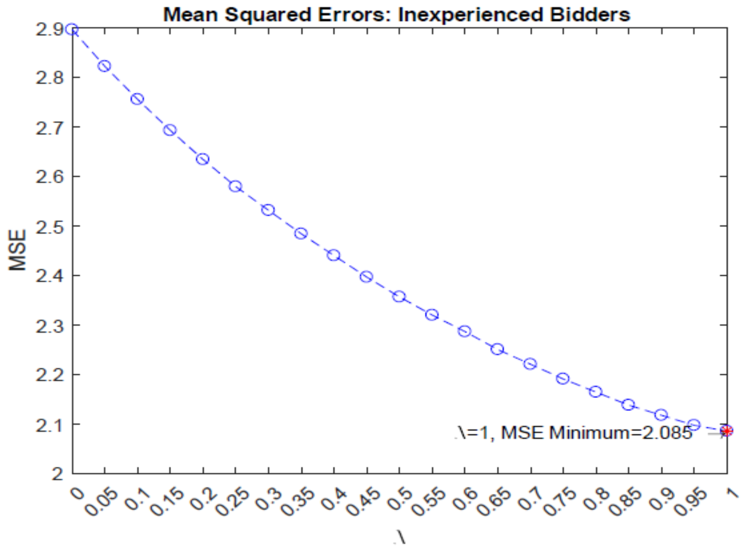

Figure 2. MSE of BNLK with Different λ: inexperienced bidders.

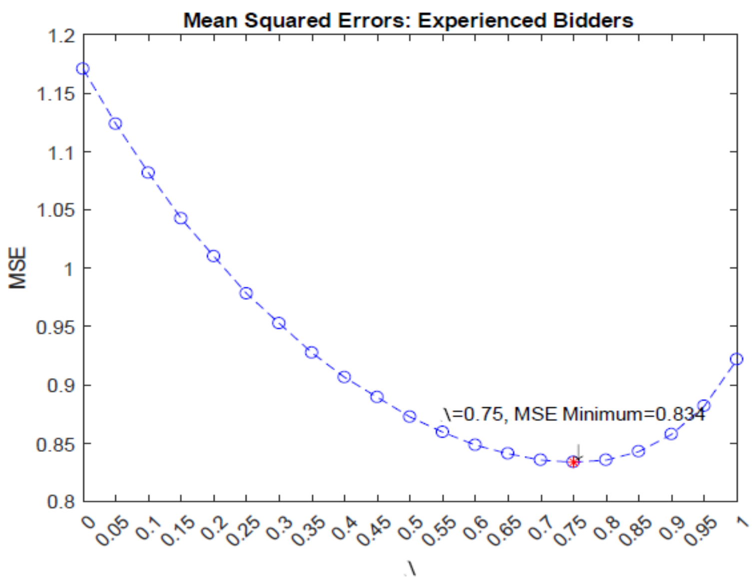

Figure 3. MSE of BNLK with different λ: experienced bidders.

6. Conclusion

This paper proposes a new solution concept, NLK that connects NE and the Level-K. It allows a player to believe that her opponent may be less, or as sophisticated, as herself, a view with support in psychology. NLK is well-defined in both static and dynamic games, making it easy to apply to the data from four published papers on static, dynamic, and auction games. In all four cases, NLK provides better predictions than those of NE and the Level-K, except for few cases when predictions coincide or when we allow the Level-K to choose freely more parameters

NLK allows having other beliefs on the naïve player and can be extended to having heterogeneous beliefs about opponents who are from distinct samples, and to games with more than two players.

Comparing across applications to experimental data, it appears that the "best fitting" λ depends on the particular game played and the population of players. Our intuition and the limited evidence that we have so far suggest that, *ceteris paribus*, we ought to expect a smaller λ in simpler games requiring less cognitive/strategical sophistication, or with more sophisticated/experienced (e.g., chess) players. For instance, in the common-value auction with inexperienced bidders, $\lambda = 1$ provides the best fit, but with experienced bidders, $\lambda = 0.75$ fits better. We also find that in the Centipede game the best λs are smaller than those in the common-value auction. It suggests that further experimental research is needed with the same, or similar, games with similar populations, to estimate the optimal λ, to provide more evidence and tests for external validity. [39]

[39] For instance, Brünner (2018) documents the consistent good performance of NLK in two similar games with the same λ: a regular rank-order tournament and the version with an outside option.

References

Alaoui, Larbi, Antonio Penta. 2016. "Endogenous Depth of Reasoning." *Review of Economic Studies* 83 (4): 1297-1333.

Arad, Ayala and Ariel Rubinstein. 2012. "The 11-20 Money Request Game: A Level-k Reasoning Study." *American Economic Review* 102(7): 3561-3573.

Avery, Christopher and John H. Kagel. 1997. "Second-Price Auctions with Asymmetric Payoffs: An Experimental Investigation." *Journal of Economics & Management Strategy* 6 (3): 573-603.

Bornstein, Gary, Tamar Kugler, and Anthony Ziegelmeyer. 2004. "Individual and Group Decisions in the Centipede Game: Are Groups More 'Rational' Players." *Journal of Experimental Social Psychology* 40 (5): 599-605.

Brünner Tobias. 2018. "Self-Selection with Non-Equilibrium Beliefs: Predicting the Outcome of Tournament Experiment." *Working Paper, Lincoln International Business School*.

Camerer, Colin F., Teck-Hua Ho, and Juin-Kuan Chong. 2004. "A Cognitive Hierarchy Model of Games." *The Quarterly Journal of Economics* 119 (3): 861-898.

Costa-Gomes, Miguel A. and Vincent P. Crawford. 2006. "Cognition and Behavior in Two-person Guessing Games: An Experimental Study." *American Economic Review* 96 (5): 1737-1768.

Costa-Gomes, Miguel A., Vincent P. Crawford, and Bruno Broseta. 2001. "Cognition and Behavior in Normal-form Games: An Experimental Study. *Econometrica* 69 (5): 1193-1235.

Crawford, Vincent P., Miguel A. Costa-Gomes, and Nagore Iriberri. 2013. "Structural Models of Nonequilibrium Strategic Thinking: Theory, Evidence, and Applications." *Journal of Economic Literature* 51 (1): 5-62.

Crawford, Vincent P. and Nagore Iriberri. 2007. "Level-k Auctions: Can a Nonequilibrium Model of Strategic Thinking Explain the Winner's Curse and Overbidding in Private-Value Auctions." *Econometrica* 75 (6): 1721-1770.

Dawes, Robyn M., 1990. "The Potential Non-falsity of the False Consensus Effect." *Insights in Decision Making: A Tribute to Hillel J. Einkorn.* Chicago: University of Chicago Press.

Engelmann, Dirk and Martin Strobel. 2000. "The False Consensus Effect Disappears If Representative Information and Monetary Incentives Are Given." *Experimental Economics* 3 (3): 241–260.

Fey, Mark, Richard D. McKelvey, and Thomas R. Palfrey. 1996. "An Experimental Study of Constant-sum Centipede Games." *International Journal of Game Theory* 25 (3): 269-287.

Fudenberg, Drew and Jean Tirole. 1993. "Perfect Bayesian Equilibrium and Sequential Equilibrium." *Journal of Economic Theory* 53 (2): 236-260.

Goeree, Jacob K., Charles A. Holt. 2001. "Ten Little Treasures of Game Theory and Ten Intuitive Contradictions." *American Economic Review* 91 (5): 1402–1422.

Goeree, Jacob K., Charles A. Holt, and Thomas R. Palfrey. 2008. "Quantal response equilibrium." *The New Palgrave Dictionary of Economics*. Palgrave Macmillan, Basingstoke.

Glicksberg, Irving L. 1952. "A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium Points." *American Mathematical Society* 3 (1): 170-174.

Ho, Teck-Hua and Xuanming Su. 2013. "A Dynamic Level-k model in Sequential Games." *Management Science* 59 (2): 452-469.

Jimenez-Gomez, David. 2018. “False Consensus in Games: Bounded Reasoning and Recursive Beliefs.” *Working paper*. https://ssrn.com/abstract=3216040.

Kagel, John H., Ronald M. Harstad, and Dan Levin. 1987. “Information Impact and Allocation Rules in Auctions with Affiliated Private Values: A Laboratory Study.” *Econometrica* 55(6): 1275-1304.

Kakutani, Shizuo. 1941. “A Generalization of Brouwer’s Fixed Point Theorem.” *Duke Mathematical Journal* 8 (3): 457-459.

Kenny, David A. 1994. *Interpersonal Perception: A Social Relations Analysis*. New York: Guilford Press.

Kreps, David M., Paul Milgrom, John Roberts, and Robert Wilson. 1982. “Rational Cooperation in the Finitely Repeated Prisoners’ Dilemma.” *Journal of Economic Theory* 27 (2): 245–252.

Kreps, David M. and Robert Wilson. 1982. “Sequential Equilibria.” *Econometrica* 50 (4): 863-894.

Kreps, David M. and Robert Wilson. 1982. “Reputation and Imperfect Information.” *Journal of Economic Theory* 27: 253-279.

Krueger, Joachim and Russell W. Clement. 1994. “The Truly False Consensus Effect: An Ineradicable and Egocentric Bias in Social Perception.” *Journal of Personality and Social Psychology* 67 (4): 596–610.

Levitt, Steven D., John A. List, and Sally E. Sadoff. 2011. “Checkmate: Exploring Backward Induction Among Chess Players.” *American Economic Review* 101 (2): 975–990.

Marks, Gary and Norman Miller. 1987. “Ten Years of Research on the False-consensus Effect: An Empirical and Theoretical Review.” *Psychological Bulletin* 102 (1): 72-90.

McKelvey, Richard D. and Thomas R. Palfrey. 1992. “An Experimental Study of the Centipede Game.” *Econometrica* 60 (4): 803-836.

McKelvey, Richard D. and Thomas R. Palfrey. 1995. “Quantal Response Equilibria for Normal Form Games.” *Games and Economic Behavior* 10 (1): 6-38.

Milgrom, Paul R. and John Roberts. 1982. “Predation, Reputation, and Entry Deterrence.” *Journal of Economic Theory* 27 (2): 280-312.

Milgrom, Paul R. and Robert J. Weber. 1982. “A Theory of Auctions and Competitive Bidding.” *Econometrica* 50 (5): 1089-1122.

Mullen, Brian, Jennifer L. Atkins, Debbie S. Champion, Cecelia Edwards, Dana Hardy, John E. Story, and Mary Vanderklok. 1985. “The False Consensus Effect: A Meta-Analysis of 115 Hypothesis Tests.” *Journal of Experimental Social Psychology* 21 (3): 262–283.

Nagel, Rosemarie. 1995. “Unraveling in Guessing Games: An Experimental Study.” A*merican Economic Review* 85 (5): 1313-1326.

Nagel, Rosemarie and Fangfang Tang. 1998. “Experimental Results on the Centipede Game in Normal Form: An Investigation on Learning.” *Journal of Mathematical Psychology* 42 (2), 356-384.

Palacios-Huerta, Ignacio, and Oscar Volij. 2009. “Field Centipedes.” *American Economic Review* 99 (4): 1619-1635.

Rapoport, Amnon and Chammah, A. M. 1966. “The Game of Chicken.” *American Behavioral Scientist* 10 (3): 10-28.

Rapoport, Amnon, William E. Stein, James E. Parco, and Thomas E. Nicholas. 2003. “Equilibrium Play and Adaptive Learning in a Three-person Centipede Game.” *Games and Economic Behavior* 43 (2): 239-265.

Rosenthal, Robert W. 1981. “Games of Perfect Information, Predatory Pricing and the Chain-store Paradox.” *Journal of Economic Theory* 25 (1): 92-100.

Ross, Lee, David Greene, and Pamela House. 1977. “The ‘False Consensus effect’: An Egocentric Bias in Social Perception and Attribution Processes.” *Journal of Experimental Social Psychology* 13 (3): 279-301.

Selten, Reinhard, 1978. “The Chain Store Paradox.” *Theory and Decision* 9 (2): 127-159.

Sherman, Steven J., Clark C. Presson, and Laurie Chassin. 1984. “Mechanisms Underlying the False Consensus Effect: The Special Role of Threats to the Self.” *Personality and Social Psychology Bulletin* 10 (1): 127–138.

Stahl, Dale O. and Paul W. Wilson. 1994. “Experimental Evidence on Players’ Models of Other Players.” *Journal of Economic Behavior and Organization* 25 (3): 309-327.

Stahl, Dale O. and Paul W. Wilson. 1995. “On Players’ Models of Other Players: Theory and Experimental Evidence.” *Games and Economic Behavior* 10 (1): 218-254.

Strzalecki, Tomasz, 2014. “Depth of Reasoning and Higher Order Beliefs.” *Journal of Economic Behavior and Organization* 108: 108-122.

Appendix A

A.1. Solve for λ-NLK equilibrium in the Money Request Game

We only go through the solution for $0 \leq \lambda < \frac{1}{2}$, since a similar argument follows for $\frac{1}{2} \leq \lambda < 1$.

We first claim that when $0 \leq \lambda < \frac{1}{2}$, \$20 must be played by an NLK player. Assume for contraction that \$20 will not be played, then deviation to \$20 would end up with \$20 for sure whereas choosing \$19 generates \$19+ $\lambda \times \frac{1}{10} \times 20 (< 20)$. So, \$19 will not be played by an NLK player. By induction, no strategy is valid for an NLK player. This is a contradiction.[40] So \$20 must be played by an NLK player. But \$20 couldn’t be the only pure strategy of an NLK player since he has an incentive to deviate to \$19. Assume that $j < 19$ is the largest number that is played with positive probability. Hence deviating to \$19 generates strictly larger payoff. Then \$19 must be played with positive probability. Denote the probability of playing \$j in the NLK equilibrium by $\beta_j, \beta_j \in [0, 1]$ and $\sum \beta_j = 1$. The expected payoff of all strategies in equilibrium should be the

[40] Since the game is finite, by Proposition 1, NLK exists.

same, and since playing \$20 yields \$20 for sure, it follows: $19 + (1-\lambda)\beta_{20}20 + \lambda\frac{20}{10} = 20$. Then $\beta_{20}^* = \frac{1-2\lambda}{(1-\lambda)20} < 1$. By the same argument, $18 + (1-\lambda)\beta_{19}20 + \lambda\frac{20}{10} = 20$. Then $\beta_{19}^* = \frac{2-2\lambda}{(1-\lambda)20}$. Because $\beta_{20}^* + \beta_{19}^* < 1$, \$18 has to be played in equilibrium (otherwise there would be an incentive to deviate to \$18), so iteratively, we get $\beta_{18}^* = \frac{3-2\lambda}{(1-\lambda)20}, \beta_{17}^* = \frac{4-2\lambda}{(1-\lambda)20}, \beta_{16}^* = \frac{5-2\lambda}{(1-\lambda)20}$. Suppose \$14 is played in equilibrium too, then $14 + (1-\lambda)\beta_{15}20 + \lambda\frac{20}{10} = 20$ implies that $\beta_{15} = \frac{6-2\lambda}{(1-\lambda)20}$. But in this case, $\sum_{j=15}^{20}\beta_j > 1$. This is a contradiction. So \$14 (and all lower numbers) would not be played by an NLK player. Then $\beta_{15}^* = 1 - \sum_{j=16}^{20}\beta_j = \frac{5-10\lambda}{(1-\lambda)20}$. In conclusion, when $0 \le \lambda < \frac{1}{2}$, there is a unique mixed-strategy for an NLK player where

$\{\sigma_i^*\} = \{\beta_{15}^*, \beta_{16}^*, \beta_{17}^*, \beta_{18}^*, \beta_{19}^*, \beta_{20}^*\} = \{\frac{5-10\lambda}{(1-\lambda)20}, \frac{5-2\lambda}{(1-\lambda)20}, \frac{4-2\lambda}{(1-\lambda)20}, \frac{3-2\lambda}{(1-\lambda)20}, \frac{2-2\lambda}{(1-\lambda)20}, \frac{1-2\lambda}{(1-\lambda)20}\}$.

A.2. Solve for λ-NLK equilibrium in the Common-Value Auction

Assume there is a linear pure strategy for a λ-NLK player, denote it as $b^\lambda(x) = b^\lambda(1) + \frac{b^\lambda(4)-b^\lambda(1)}{3}(x-1), x \in [1,4]$. Denote $d^\lambda = b^\lambda(4) - b^\lambda(1)$.The probability that the opponent is level$_0$ conditional on a tie is $q^\lambda = \Pr(rival = level_0 | tie\ at\ bid = b) = \frac{\lambda/6}{\lambda/6+(1-\lambda)/d^\lambda}, b \in [b^\lambda(1), b^\lambda(4)] \subseteq [2,8]$. Then by indifference in the case of Maximum Willingness to Pay conditional on a tie, denoted by $MWP(X) = b(x)$:

MWP(1)= $q^\lambda(1+2.5) + (1-q^\lambda)2 = 1.5q^\lambda + 2 = b^\lambda(1)$,

MWP(4)= $q^\lambda(4+2.5) + (1-q^\lambda)8 = 8 - 1.5q^\lambda = b^\lambda(4)$.

Then $d^\lambda = b^\lambda(4) - b^\lambda(1) = 1 - 3q^\lambda = \frac{\lambda/6}{\lambda/6+(1-\lambda)/d^\lambda}$.

Then $\left(d^\lambda\right)^2 + 3\frac{2-3\lambda}{\lambda}d^\lambda - \frac{36(1-\lambda)}{\lambda} = 0$.

So, the bidding strategy is $b^{\lambda}(x) = b^{\lambda}(1) + \frac{d^{\lambda}}{3}(x-1)$, where $b^{\lambda}(1) = 1.5q^{\lambda} + 2$, $d^{\lambda} = \frac{3}{2\lambda}(3\lambda + \sqrt{-7\lambda^2 + 4\lambda + 4} - 2)$ and $q^{\lambda} = (1 - d^{\lambda})/3$.

A.3. Comparing Other Related Literature

Eyster and Rabin (2005), proposed Cursed Equilibrium (CE) that also relaxes the restriction on beliefs in NE, while maintaining the equilibrium concepts for players' strategies. They show that CE rationalizes behavior (data) from experiments where BNE fails. In particular, this rationalization occurs in Common-Value Auctions, where Kagel and Levin (2002) found systematic overbidding and losses, a phenomenon called the Winner's Curse. CE also fits experimental data from voting and signaling models better than BNE. In one extreme version of CE, entitled as "fully CE," people correctly predict other players' distribution of actions, but ignore the correlation between actions and the specific players' types who chose those actions. In their general model, χ-CE, beliefs are a weighted average of beliefs in fully cursed opponents (with weight χ) and Bayesian Nash opponents (with weight $(1-\chi)$). The CE characterizes heterogeneous behaviors by different cursed levels (with $\chi = 1$ being fully cursed, and $\chi = 0$ being BNE). However, CE reduces to NE when there is complete information. Hence it cannot be applied to explain deviations from NE in both static and dynamic games with complete information. Conceptually, the CE models bounded rational agents as only partially taking into account how other players' actions depend on their type. In contrast, NLK allows a player to consider the possibility that the other player is a naive player or another NLK player like herself. Kets (2017) extends the type space tracing back to Harsanyi (1976) by allowing players to have finite instead of infinite depth of reasoning. However, different from NLK, it required that the depth of reasoning is the same for all players and beliefs that the other player might be less sophisticated is not allowed.

For application to dynamic games with perfect information and recall, Analogy-Based Expectation Equilibrium (ABEE), a solution concept proposed by Jehiel (2005), is the most closely related to ours.[41] In ABEE, agents first group the set of opponents' decision nodes into a partition, namely, an analogy class. Then, they form expectations about each opponent's average behavior

[41] Jehiel and Koessler (2008) extends his analogy-based concept to Bayesian games.

at every element of the analogy class rather than, more precisely, at each decision node. Though conceptually, ABEE is similar to CE, when applied to a different type of games, ABEE also suggests that people might not fully consider how others' choices depend on their information, and such deficiency in reasoning is common knowledge among all players.[42] In contrast, our model allows NLK players to consider heterogeneity in their opponents' inference process. In our adaptation of NLK equilibrium to dynamic games, beliefs for different types of opponents are anchored at the beginning of the game and are updated at each stage using Bayes' Rule. Analytically, ABEE coincides with SPNE for the finest analogy partition and as NLK, ABEE can also rationalize *Passing*, in the Centipede Game, to the last few stages for a large range of partitions, in violation of the backward induction predictions. However, ABEE does not provide a specific way to choose an analogy class, while NLK offers a way of parametric estimation to specify beliefs in equilibrium.[43]

As all of the aforementioned solution concepts, NLK maintains the best response to beliefs but relaxes NE's requirement of player' consistent beliefs about other players. In dynamic games, Aumann (1992), like several other writers afterward, has shown that a failure of backward induction does not imply a failure of individual rationality. For example, in the Centipede Game, backward induction implies that the first mover ought to use Stop at the first decision node, which is rarely found in experimental data. These papers show that some relaxations of the "common knowledge of rationality," can explain several rounds of Passing although all of the players are individually rational.[44,45]

[42] More specifically, information means the history upon reaching a decision node at which the choice is made.

[43] As an extension of the Level-K model to dynamic games, Ho and Su (2013) apply their model to the experiment data of the Centipede Game. However, they intend to study learning across repetitions while ours explains strategic behavior better even for novel games. Moreover, unlike their model, NLK does not restrict the strategy set, which allows NLK to capture Bayesian updating for beliefs across stages within one round.

[44] One must be careful about the terminology according to the epistemic condition of NE. Aumann and Brandenburger (1995) prove that in a two persons' game, mutual knowledge of preferences and payoffs, rationality, and beliefs for the other players' strategies are sufficient for NE. This is, common knowledge of rationality is not necessary for NE in two persons' game. Moreover, Battigalli and Bonanno (1999) argue that there is a contradiction between results of backward induction and a common belief in sequential rationality at later stages. Thus, in this paper by "common knowledge of rationality," we mean in general, the extra assumptions needed for NE/BNE/SPNE besides individual rationality.

[45] Aumann (1992) shows that continuation of the game beyond the first node for several rounds could occur even with "mutual knowledge" of high degrees. Considering the fact that some sequentially rational behaviors off the equilibrium path are only reachable by the violation of sequential rationality, Reny (1992) defines a weaker version of sequential rationality in light of forward induction. Ben-Porath (1997) proves that cooperation in the Centipede Game is consistent with Common Certainty of Rationality, a weaker concept than the Common Knowledge of Rationality. Asheim and Dufwenberg (2003) introduce the concept of "Fully Permissible Sets" to the extensive form

Reference for supplementary literature:

Asheim, Geir B. and Martin Dufwenberg. 2003. "Deductive Reasoning in Extensive Games." *Economic Journal* 113 (487): 305-325.

Aumann, Robert J. 1992. "Irrationality in Game Theory." *Economic analysis of markets and games: Essays in honor of Frank Hahn*: 214-27.

Aumann, Robert J. and Adam Brandenburger. 1995. "Epistemic Conditions for Nash Equilibrium." *Econometrica* 63 (5): 1161.

Battigalli, Pierpaolo and Giacomo Bonanno. 1999. "Recent Results on Belief, Knowledge and the Epistemic Foundations of Game Theory." *Research in Economics* 53 (2): 149-225.

Ben-Porath, Elchanan. 1997. "Rationality, Nash Equilibrium and Backwards Induction in Perfect Information Games." *Review of Economic Studies* 64 (1): 23-46.

Bosch-Domenech, Antoni, Jose G. Montalvo, Rosemarie Nagel, and Albert Satorra. 2002. "One, Two, (Three), Infinity...: Newspaper and Lab Beauty Contest Experiments." *American Economic Review* 92 (5): 1687-1701.

Eyster, Erik and Matthew Rabin. 2005. "Cursed Equilibrium." *Econometrica* 73 (5): 1623-1672.

Harsanyi, John C., 1973. "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-strategy Equilibrium Points." *International Journal of Game Theory* 2 (1): 1-23.

Jehiel, Philippe. 2005. "Analogy-based Expectation Equilibrium." *Journal of Economic Theory* 123 (2): 81-104.

Jehiel, Philippe and Frederic Koessler. 2008. "Revisiting Games of Incomplete Information with Analogy-based Expectations." *Games and Economic Behavior* 62 (2): 533-557.

Kagel, John H. and Dan Levin. 2002. *Common Value Auctions and The Winner's Curse*. Princeton, NJ: Princeton University Press.

Kawagoe, Toshiji and Hirokazu Takizawa. 2012. "Level-k Analysis of Experimental Centipede Games." *Journal of Economic Behavior & Organization* 82 (2): 548-566.

Kets, Willemien. 2017. "Bounded Reasoning and Higher-Order Uncertainty." *Working Paper*. http://wkets.org/wp-content/uploads/2017/08/bounded_reasoning_2017-0801.pdf

Reny, Philip J. 1992. "Rationality in Extensive-form Games." *The Journal of Economic Perspectives* 6 (4): 103-118.

game, where players reason deductively by trying to figure out one another's moves. They show that deductive reasoning does not necessarily imply backward induction.